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Optimal 2-D Interleaving for Robust Multimedia Transmission

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Abstract

Interleaving is a process to rearrange code symbols so as to spread bursts of errors over multiple codewords that can be corrected by random error-correction codes. By converting burst errors into random-like errors, interleaving thus becomes an effective means to combat errors. In this paper, we focus on how to obtain effective interleaving schemes for 2D arrays, namely, how to spread the arbitrary error burst such that they are separated as far as possible. To achieve this, the theoretical bound for optimal 2D interleaving on arbitrary sized 2D array is analyzed. Based on it, a novel sphere tiling based method is proposed to achieve this bound. We first present this method for a specified square array, then we extend to arbitrary sized 2D array. The validity of the proposed method is proven by showing gains in multimedia transmission.

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otherwise stated.

Definition 2.1: L^1-distance in Z^M. Let $x, x' \in Z^M$, $x = (x_1, x_2, \ldots, x_M)$, $x' = (x'_1, x'_2, \ldots, x'_M)$. Then we define

$$\begin{aligned} d(x,x') &= \|x - x'\|_1 \\ &= |x_1 - x'_1| + |x_2 - x'_2| + \cdots + |x_M - x'_M|. \end{aligned}$$

Two elements x and x' are called neighbors if and only if $d(x,x') = 1$.

In an M-D array, an element x has a total of $2M$ neighbors. In 2-D arrays, these neighbors are given by $(x_1 \pm 1, x_2)$ and $(x_1, x_2 \pm 1)$. In 3-D arrays, the neighbors of (x_1, x_2, x_3) are given by $(x_1 \pm 1, x_2, x_3)$, $(x_1, x_2 \pm 1, x_3)$, and $(x_1, x_2, x_3 \pm 1)$.

Definition 2.2: Let C be an M-D code of $m_1 \times m_2 \times \cdots \times m_M$ over Z_q. A codeword of C is an M-D array of $m_1 \times m_2 \times \cdots \times m_M$, with each element of the M-D array assigned with a code symbol.

Note that Z_q denotes the ring with elements $0, 1, \ldots, q-1$. When q is prime, Z_q becomes the Galois field $GF(q)$.

Definition 2.3: In M-D arrays, a burst B is a connected subset of the given M-D array in which any element has at least one neighbor contained in B. The size of B is defined as the number of elements in B.

Interleaving generally means mixing up code symbols so that each element in an error burst can be spread into different codewords (with respect to one random error-correction codes). Therefore, if any two elements within a distinct codeword in the 2-D interleaved array are separated as far as possible from one another in the de-interleaved array, then a big error burst can be corrected.

Let A be an M-D array of $m_1 \times m_2 \times \cdots \times m_M$ and $x = (x_1, x_2, \ldots, x_M) \in A$ with $0 \leq x_i < m_i, 1 \leq i \leq M$. We re-index each element x of A as S_k with $k = k(x)$ uniquely determined by x. For example, in 2-D, we can have $k = x_1 + m_1 x_2$; and in 3-D, we can use $k = x_1 + m_1 x_2 + m_1 m_2 x_3$. After having 1-D re-indexing, we consider a division of A into L blocks with each block containing $K = N/L$ elements where $N = m_1 \times m_2 \times \cdots \times m_M$.

Definition 2.4: All elements belonging to the same block are referred to as K-equivalent. That is, $S_k \approx S_l$ if and only if $\lfloor k/K \rfloor = \lfloor l/K \rfloor$.

According to Definition 2.4, we see that S_{2l} and S_{2l+1} are 2-equivalent while S_{3l}, S_{3l+1}, and S_{3l+2} are 3-equivalent. It is obvious that any K_1-equivalent elements are also K_2-equivalent if K_2 is a multiple of K_1. Let each block represent a distinct codeword with length K, then all elements within the same codeword are K-equivalent. Hence, the objective of effective interleaving is transferred to the problem of maximizing the minimum distance between any two K-equivalent elements.

Definition 2.5: Let A be an interleaved array of $m \times m$. The interleaving distance of A is defined as the shortest distance between any m-equivalent elements in A.

Definition 2.6: An $m \times m$ interleaving array A is called an *optimal interleaving array* if the interleaving distance of A attains the maximum, that is, there is no $m \times m$ array that has a larger interleaving distance than that of A.

Example 2.1: In a 2×2 array, the maximum possible distance between any two elements is bounded by 2. The following array in Fig. 1 is clearly an optimal interleaving array with maximum interleaving distance 2 between the 2-equivalent elements S_0 and S_1, and S_2 and S_3.

S_0	S_2
S_3	S_1

Fig. 1. A 2×2 optimal interleaving array.

Example 2.2: The first 5×5 array in Fig. 2 has interleaving distance 2 while the second and the third 5×5 arrays in Fig. 2 each has an interleaving distance of 3. The first is clearly not optimally interleaved. It will be clear from Theorem 3.1 below that both the seccond and the third arrays in Fig. 2 are optimally interleaved.

S_0	S_5	S_{10}	S_{15}	S_{20}
S_{21}	S_1	S_6	S_{11}	S_{16}
S_{17}	S_{22}	S_2	S_7	S_{12}
S_{13}	S_{18}	S_{23}	S_3	S_8
S_9	S_{14}	S_{19}	S_{24}	S_4

(a)

S_0	S_5	S_{10}	S_{15}	S_{20}
S_{16}	S_{21}	S_1	S_6	S_{11}
S_7	S_{12}	S_{17}	S_{22}	S_2
S_{23}	S_3	S_8	S_{13}	S_{18}
S_{14}	S_{19}	S_{24}	S_4	S_9

(b)

S_0	S_5	S_{10}	S_{15}	S_{20}
S_{11}	S_{16}	S_{21}	S_1	S_6
S_{22}	S_2	S_7	S_{12}	S_{17}
S_8	S_{13}	S_{18}	S_{23}	S_3
S_{19}	S_{24}	S_4	S_9	S_{14}

(c)

Fig. 2. Examples of 5×5 interleaving arrays. The array in (a) has interleaving distance 2 and is not optimally interleaved. Both arrays in (b) and (c) are optimally interleaved with maximum interleaving distance 3.

III. Theoretical Analysis

In order to find an optimal interleaving array, it is necessary to know the maximum (or at least an upper bound) of the possible interleaving distances for a given 2-D (or M-D) array. Assuming there is a 2-D (or M-D) array with interleaving distance d, then a set of equal sized spheres (with diameter d) can be built centered at each equivalent element without overlapping. In this section, we will present a thorough study of these 2-D spheres and the corresponding bound of interleaving distance.

The 2-D sphere $S_{2,d}$ was first studied in [8] for d odd. The idea was then extended to the case of d even in [4]. The 2-D sphere $S_{2,d}$ with diameter d under the L^1 distance in Z^2 can be defined as follows.

Definition 3.1: 2-D Spheres. Let d be a positive integer. Then we define for d odd

$$S_{2,d} = \{x \in Z^2 : |x_1| + |x_2| < d/2\};$$

and for d even

$$S_{2,d} = \{x \in Z^2 : |x_1 - 1/2| + |x_2| < d/2\}.$$

Some typical 2-D spheres are shown in Fig. 3.

Lemma 3.1: $S_{2,d}$ defines a 2-D sphere with diameter d (centered at $(0,0)$ if d is odd, and at $(0,0)$ and $(1,0)$ if d is even). For any $x \in S_{2,d}$, $y \in S_{2,d}$, it holds $d(x,y) < d$; and for any $x \notin S_{2,d}$, there exists $y \in S_{2,d}$ such that $d(x,y) \geq d$.

It is clear that the sphere $S_{2,d}$ can be embedded in a $d \times d$ square array. Further, we note that for d odd, the 2-D sphere $S_{2,d}$ defined above is symmetric with respect to the center $(0,0)$. For d even, $S_{2,d}$ is centered at $(0,0)$ and $(1,0)$ and is no long symmetric with respect to $(0,0)$, instead it has a long axis along the x_1 direction. Geometrically, $S_{2,d}$ can be constructed recursively by appending all neighbors of $S_{2,d-2}$, starting with a single element $(0,0)$ if d is odd, or two neighboring elements $(0,0)$ and $(1,0)$ if d is even.

Counting the elements in $S_{2,d}$, one can show that

Lemma 3.2: The 2-D sphere $S_{2,d}$ contains $(d^2+1)/2$ elements if d is odd, and $d^2/2$ elements if d is even. In other words, we have for all $d \geq 1$,

$$|S_{2,d}| = \lceil d^2/2 \rceil = \begin{cases} (d^2+1)/2 & \text{if } d \text{ is odd} \\ d^2/2 & \text{if } d \text{ is even} \end{cases}$$

Theorem 3.1: The maximal interleaving distance in an $m \times m$ array is bounded by $d = \lfloor \sqrt{2m} \rfloor$.

Proof: We assume $m > 2$ as the case for $m = 1$ or $m = 2$ is trivial. Then we have $\sqrt{2m} < m$, and hence $d = \lfloor \sqrt{2m} \rfloor \leq m-1$. Consider a sphere $S_{2,d+1}$ with diameter $d+1$ that is embedded in the $m \times m$ array. Suppose on the contrary there exists an interleaving for the $m \times m$ array with an interleaving distance $\geq d+1$. Since the distance between any elements in $S_{2,d+1}$ is always less than $d+1$ (see Lemma 3.1), each element in $S_{2,d+1}$ has to belong to a different codeword. Therefore, $|S_{2,d+1}| \leq m$. Using Lemma 3.2, we obtain, $(d+1)^2/2 \leq |S_{2,d+1}| \leq m$, that is, $d+1 \leq \sqrt{2m}$. This contradicts $d = \lfloor \sqrt{2m} \rfloor > \sqrt{2m} - 1$. ∎

IV. Sphere tiling based 2-D interleaving method

By using the 2-D spheres to tile a 2-D array, the size of the array should be a multiple of the size of the 2-D sphere. To generate an interleaving array that satisfies the above upper bound, one needs to find a way to tile the 2-D spheres such that no overlapping and space exist in the $m \times m$ array. Due to these restrictions, it can be conjectured that the distance between two co-positional elements of two neighboring spheres should be the bound d. In the following, we will propose some tiling methods. We first use two examples to show the principle. Then we generalize them by using mathematical analysis. The validity is proved afterwards.

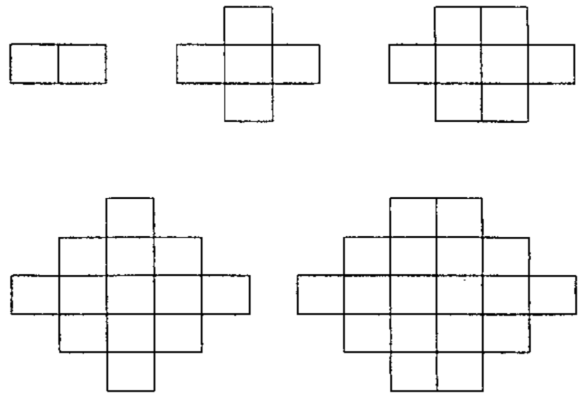

Fig. 3. Typical 2-D sphere with size 2, 5, 8, 13, 18.

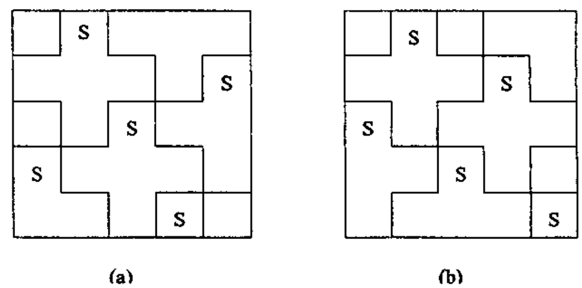

Fig. 4. Tiling of size 5 sphere: (a) Tiling by translation $(x+2, y+1)\,(\mathrm{mod}\ m)$. (b) Tiling by translation $(x+1, y+2)\,(\mathrm{mod}\ m)$.

Figure 4 shows the example for the case of $d = 3$, where S denotes top element of each sphere. Given S with position $(0,1)$, its nearest co-positional element can be located in either $(1,3)$ or $(2,2)$. Continue the tiling procedure and let the position value be modulo m, the interleaved arrays can be obtained as Figure 4 (a) and (b) respectively.

Figure 5 shows the example for the case of $d = 4$. Similarly, given S with position $(0,1)$, its neighboring co-positional element can be located in $(1,4)$, $(2,3)$, $(3,0)$ or $(3,2)$. The corresponding interleaved arrays are illustrated in Figure 5 (a) and (b) respectively. Note that the choice of $(1,4)$ produces the same tiling structure as that $(3,2)$. This is also true for $(2,3)$ and $(3,0)$.

After investigation, we find that if d is odd, given S_i with position (x,y), its neighboring co-positional elements S_j can be located at

$$\begin{cases} (x+(d+1)/2, y \pm (d-1)/2)\,(\mathrm{mod}\, m), \text{ or} \\ (x+(d-1)/2, y \pm (d+1)/2)\,(\mathrm{mod}\, m). \end{cases}$$

If d is even, the neighboring co-positional elements S_j can be located in any of the following positions

$$\begin{cases} (x+d/2-1, y \pm (d/2+1))\,(\mathrm{mod}\, m), \text{ or} \\ (x+d/2+1, y \pm (d/2-1))\,(\mathrm{mod}\, m), \text{ or} \\ (x+d/2, y \pm d/2)\,(\mathrm{mod}\, m). \end{cases}$$

Continuing the above tiling procedure, we can see that there will be only one co-positional element of S_i in each row of the

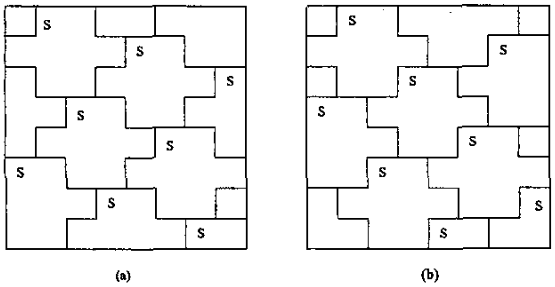

Fig. 5. Tiling of size 8 sphere: (a) Tiling by translation $(x+1, y+3) \pmod m$ or $(x+3, y+1) \pmod m$. (b) Tiling by translation $(x+2, y+2) \pmod m$ or $(x+3, y-1) \pmod m$.

array. Motivated by this observation, we propose the optimal interleaving array construction methods for m equal to the size of the 2-D sphere as the following.

Procedure 4.1: Let A be a 2-D array of $m \times m$ with $m = |S_{2,d}|$, $d = \lfloor\sqrt{2m}\rfloor$. We define, for d odd,

$$b = \begin{cases} d & \text{translations by } ((d+1)/2, (d-1)/2) \text{ or } \\ & ((d-1)/2, -(d+1)/2), \\ -d & \text{translations by } ((d-1)/2, (d+1)/2) \text{ or } \\ & ((d+1)/2, -(d-1)/2) \end{cases}$$

and for d even,

$$b = \begin{cases} d-1 & \text{translations by } (d/2+1, d/2-1), \\ d+1 & \text{translations by } (d/2, d/2), \\ -(d-1) & \text{translations by } (d/2+1, -(d/2-1)), \\ -(d+1) & \text{translations by } (d/2, -d/2) \end{cases}$$

Then we construct an $m \times m$ interleaving array by using the following recipe

$$(0, j) \mapsto (0, j) \text{ and } (i, j) \mapsto (i, (ib+j) \pmod m)).$$

By using this procedure to construct 5×5 and 8×8 interleaving arrays, the results exactly match the sphere tilings in Figure 4 and Figure 5.

Example 4.1: Consider the case of 5×5 array ($m = 5$). Then we have $m = |S_{2,d}|$, $d = 3$ and $b = 3$. Using the above procedure, we obtain the 5×5 optimal interleaving array in Fig. 2(c). Note that, by symmetry, the choice $b = 2 \equiv -3 \pmod m$ also generates a 5×5 optimal interleaving array in this case, see Fig. 2(b).

In the following, we will discuss the optimality of Procedure 4.1. For brevity, we will only consider the case $b = d$ (for d odd) and $b = d-1$ (for d even), that is,

$$b = \begin{cases} d & \text{if } d \text{ is odd} \\ d-1 & \text{if } d \text{ is even} \end{cases}$$

Lemma 4.1: For d odd and $m = (d^2+1)/2$, then $b = d$ is relative prime to m. For d even and $m = d^2/2$, then $b = d-1$ is relative prime to m. That is, $\gcd(b, m) = 1$.

Define

$$\xi_i = bi \pmod m, \quad 0 \le i < m.$$

Then the method in Procedure 4.1 is a cyclic translation

$$(i, j) \mapsto (i, (\xi_i + j) \pmod m))$$

For the case $b = d$ for d odd and $b = d-1$ for d even, we have the following result.

Lemma 4.2: For all $i \ne k$, $0 \le i, k < m$, it holds

$$|i-k| + |\xi_i - \xi_k| \ge d, \quad |i-k| + m - |\xi_i - \xi_k| \ge d.$$

Theorem 4.1: The interleaving array constructed by Procedure 4.1 with $b = d$ for d odd, $b = d-1$ for d even and $m = |S_{2,d}|$ is an optimal interleaving array.

Proof: Consider two equivalent elements $P = (i, \xi_i + j \pmod m))$ and $Q = (k, \xi_k + j \pmod m))$. Since $\xi_i + j \pmod m) - (\xi_k + j \pmod m)) = \xi_i - \xi_k \pmod m)$ and $-m < \xi_i + j \pmod m) - (\xi_k + j \pmod m)) < m$, we have either $|\xi_i + j \pmod m) - (\xi_k + j \pmod m))| = |\xi_i - \xi_k|$ or $|\xi_i + j \pmod m) - (\xi_k + j \pmod m))| = m - |\xi_i - \xi_k|$. From Lemma 4.2, we have $d(P, Q) = |i-k| + |\xi_i + j \pmod m) - (\xi_k + j \pmod m))| \ge d$. Theorem 4.1 now follows. ■

Note that for d odd and $b = d$, we have $b(d+1)/2 \equiv (d-1)/2 \pmod m) = \xi_{(d+1)/2}$ and $b(d-1)/2 \equiv -(d+1)/2 \pmod m) = \xi_{(d-1)/2}$. Similarly for d even and $b = d-1$, we have $b(d/2+1) \equiv d/2 - 1 \pmod m) = \xi_{d/2+1}$. Finally we observe that by symmetry ($\xi_i \mapsto -\xi_i$), the same proof above also shows that the interleaving array constructed by Procedure 4.1 with $b = -d$ for d odd and $b = -(d-1)$ for d even is also an optimal interleaving array. We leave it to the reader to check the validity of the choice $b = \pm(d+1)$ (for d even only).

In the above, we have presented a sphere tiling based method for constructing optimal square interleaving array. In the following, we first present some theoretical analysis for arbitrary size square interleaved array, then we propose a method to generate the interleaved array that reaches the upper bound.

Notice that the size of the 2-D sphere is not consecutive. For most of the integer m such as $3, 4, 6, 7, 9, 10, 11, \cdots$, there exists no sphere with size m. Consider $m \times m$ array. Let $d = \lfloor\sqrt{2m}\rfloor$. We assume $m_1 = |S_{2,d+1}| > m > m_0 = |S_{2,d}|$. Then we have the following result.

Lemma 4.3: For any positive integer m, the maximum interleaving distance in an $m \times m$ array is less than or equal to $d = \lfloor\sqrt{2m}\rfloor$.

According to Lemma 4.3, the upper bound of the minimum distance is the diameter of the largest sphere with size less than or equal to m. According to this observation, the upper bound is 2 for $2 \le m \le 4$, 3 for $5 \le m \le 7$, 4 for $8 \le m \le 12$, etc. In the following, we propose a method to construct, for arbitrary m, an $m \times m$ optimal interleaving array that achieves the above sphere packing upper bound.

For arbitrary $m \times m$ square arrays, we have the following generalization of Theorem 4.1.

Theorem 4.2: Let m be any positive integer and define

$$d = \lfloor\sqrt{2m}\rfloor, \quad m_0 = |S_{2,d}|, \quad \xi_i = bi \pmod{m_0},$$

S_0	S_3	S_6
S_7	S_1	S_4
S_2	S_5	S_8

Fig. 6. Illustration of 3×3 interleaved array by using the algorithm in Theorem 4.2.

$$b = \begin{cases} d & \text{if } d \text{ is odd,} \\ d-1 & \text{if } d \text{ is even.} \end{cases}$$

Then the following construction

$$(i, j) \mapsto (i, (\xi_i + j) \pmod{m})), \quad 0 \le i, j < m.$$

generates an optimal interleaving array with interleaving distance $\ge d = \lfloor \sqrt{2m} \rfloor$.

Figure 6 shows an example of 3×3 array generated by using the algorithm in Theorem 4.2.

Figure 7 shows an example of 12×12 array that generated by using the above algorithm. Three sets of equivalent elements $\{S_0, \cdots, S_{11}\}$, $\{S_{72}, \cdots, S_{83}\}$ and $\{S_{132}, \cdots, S_{143}\}$ are demonstrated.

S_0						S_{72}					S_{132}
		S_{133}	S_1						S_{73}		
S_{74}					S_{134}	S_2					
S_{135}	S_3						S_{75}				
			S_{136}	S_4						S_{76}	
	S_{77}					S_{137}	S_5				
	S_{138}	S_6						S_{78}			
				S_{139}	S_7						S_{79}
S_8						S_{80}					S_{140}
		S_{141}	S_9						S_{81}		
S_{82}					S_{142}	S_{10}					
S_{143}	S_{11}						S_{83}				

Fig. 7. Illustration of 12×12 interleaved array by using the algorithm in Theorem 4.2.

V. Summary

In this paper, we address the protection of multimedia data by applying optimal interleaving on them, specifically, 2-D digital data. To do so, the theoretical characteristics of M-D interleaving is analyzed based on the concept of interleaving distance. We first present and proved the upper bound of the interleaving distance for 2-D array. Then a novel 2-D interleaving method is proposed by the motivation of sphere tiling. By using the proposed method, each element in the same size m codeword can be spreaded into different codewords in the $m \times m$ array. Equivalently, any error burst of size m can be spreaded into to different code blocks in the array. Thus, the simple error correction code which is optimal for independent channel can be used to correct this kind of error bursts.

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