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TR2005-099 November 2005

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IEEE Globecom 2005

A Flexible Lognormal Sum Approximation Method

Jingxian Wu[†*], *Member, IEEE,* Neelesh B. Mehta[‡], *Member, IEEE,* Jin Zhang[‡], *Senior Member, IEEE*

***Abstract*— A simple and novel method is presented to approximate the distribution of the sum of independent, but not necessarily identical, lognormal random variables, by the lognormal distribution. It is shown that matching a short Gauss-Hermite approximation of the moment generating function of the lognormal sum with that of the lognormal distribution leads to an accurate lognormal sum approximation. The advantage of the proposed method over the ones in the literature, such as the Fenton-Wilkinson method, Schwartz-Yeh method, and the recently proposed Beaulieu-Xie method, is that it provides the parametric flexibility to handle the inevitable trade-off that needs to be made in approximating different regions of the probability distribution function. The accuracy is verified using extensive simulations based on a cellular layout.**

I. INTRODUCTION

The lognormal distribution arises in several wireless systems such as cellular mobile communication systems [1, Chp. 3] and ultra wide band transmission [2]. For example, it models the attenuation due to shadowing in wireless channels. Therefore, one often encounters the sum of lognormal random variables (RV) in analyzing wireless system performance. Given the importance of the lognormal sum distribution, considerable efforts have been devoted to analyze its statistical properties. While exact closed-form expressions for the lognormal sum probability density function (pdf) are unknown, several analytical approximation methods exist in the literature [3]–[8].

The methods proposed in the literature can be classified into two broad categories. The methods by Fenton-Wilkinson [3], Schwartz-Yeh [4], and Beaulieu-Xie [6] approximate the lognormal sum by a single lognormal RV, and provide different recipes for determining the parameters of the lognormal pdf. The proven permanence of the lognormal pdf when the number of summands approaches infinity lends credence to these methods [5], [9]. The methods by Farley [1], [4], Ben Slimane [7], and Schleher [8] instead compute a compound distribution or specify it implicitly. For example, the first two methods derive strict lower bounds of the cumulative distribution function (CDF), while the last one partitions the lognormal RV's range into three segments, with each segment being approximated by a different lognormal RV.

Beaulieu *et al.* [6], [10] have studied in detail the accuracy of several of the above methods, and shown that each method has its own advantages and disadvantages; none is unquestionably better than the others. Farley's method and, more generally, the formulae derived in [7] are strict bounds that can be quite loose for certain typical parameters. The methods also differ considerably in their complexity. Only the Fenton-Wilkinson method offers closed-form solution for the underlying parameters of the approximating lognormal pdf.

Motivated by the interpretation of the moment generating function (MGF) as a weighted integral of the pdf, we present a flexible lognormal sum approximation method that matches the approximation of the MGF of the lognormal sum with that of a single lognormal RV to derive the latter's parameters. As elaborated later, the weight function can be adjusted to emphasize the accuracy in approximating different portions of the lognormal sum pdf. Moreover, the MGF of the sum of independent RVs can be easily calculated from the MGFs of the individual RVs. The proposed method uses an approximate Gauss-Hermite expansion of the lognormal MGF, and circumvents the requirement for very precise numerical computations. It is not recursive; it is numerically stable and accurate; and it offers considerable flexibility compared to previous approaches.

As mentioned, the MGF and the characteristic function (CF) possess the desirable property that the MGF (CF) of a sum of independent RVs is the product of the MGFs (CFs) of the individual RVs [11].[1] This property of the CF has also been exploited by Barakat [5] and Beaulieu-Xie [6] to numerically evaluate the lognormal sum pdf. However, their methods require very accurate numerical computation of the characteristic function because of the oscillatory property of the Fourier integrand as well as the slow decay rate of the lognormal pdf tail [6].

Barakat numerically computed the CF of the lognormal distribution using Taylor series expansion, and then applied the inverse Fourier transform to the product of lognormal CFs to determine the lognormal sum pdf. Also, no effort was made to find the analytical expressions of the approximate distribution. A similar approach was also suggested by Anderson [12]. Beaulieu-Xie's elegant and conceptually simple method first numerically evaluates the lognormal sum CDF, to a high degree of accuracy, at several points, using a modified Clenshaw-Curtis method. The composite CDF is obtained by numerically calculating the inverse Fourier transform, and is plotted on 'lognormal paper'. The parameters of the approximating lognormal distribution, which is a straight line on lognormal paper, are determined by minimizing the maximum error in a given interval.

The paper is organized as follows: Section II reviews the lognormal sum approximation methods in the literature

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[*]This work was done when the author was at MERL.

[1]While the CF can be considered a special case of the MGF, we choose to treat the two as separate to keep the discussion clear.

and investigates the reasons behind their observed behaviors. Section III motivates and defines the method proposed in this paper. Numerical examples based on a cellular layout are used in Section IV to validate the proposed method and to compare it with other methods. The conclusions follow in Section V.

II. Comparison of Various Lognormal Sum Approximation Methods

Let $X_1, \ldots, X_K$ be K independent, but not necessarily identical, lognormal RVs with pdfs, $p_{X_i}(x)$, for $1 \le i \le K$. Then each X_i can be written as $10^{0.1 Y_i}$ such that Y_i is a Gaussian random variable with mean, μ_{Y_i} dB, and standard deviation, σ_{Y_i} dB, *i.e.*, $Y_i \sim \mathcal{N}(\mu_{Y_i}, \sigma_{Y_i}^2)$.

General closed-form expressions for the pdf or CDF of the lognormal sum $\sum_{i=1}^K X_i$ are not available. However, the lognormal sum can be well approximated by a new lognormal RV $X = 10^{0.1Y}$, where Y is a Gaussian RV with mean μ_Y and variance σ_Y^2. Thus, the problem is now equivalent to determining the lognormal moments μ_Y and σ_Y^2 given the statistics of the lognormal RVs X_i, for $i = 1, \ldots, K$.

The Fenton-Wilkinson (F-W) method computes μ_Y and σ_Y^2 by exactly matching the first and second central moments of X with that of $\sum_{i=1}^K X_i$:

$$\int_0^\infty x p_X(x) dx = \sum_{i=1}^K \int_0^\infty x p_{X_i}(x) dx, \tag{1a}$$

$$\int_0^\infty (x - \mu_X)^2 p_X(x) dx = \sum_{i=1}^K \int_0^\infty (x - \mu_{X_i})^2 p_{X_i}(x) dx, \tag{1b}$$

where μ_X and μ_{X_i} are the means of X and X_i, respectively. If the K lognormal RVs are identically distributed, then the approximating lognormal moments μ_Y and σ_Y can even be expressed in closed-form. While the F-W method accurately models the *tail portion* (large values of X) of the lognormal sum pdf, it is quite inaccurate near the *head portion* (small values of X) of the sum pdf, especially for large values of σ_{Y_i} [10]. The mean square error in μ_Y and σ_Y increases with a decrease in the spread of the mean values or an increase in the spread of the standard deviations of the summands [13]. Also, in modeling the behavior of $10 \log_{10} \left(\sum_{i=1}^K X_i \right)$ the method breaks down when $\sigma_{Y_i} > 4$ dB [1].

The Schwartz-Yeh (S-Y) method instead matches the moments in the log-domain, *i.e.*, it equates the first and second central moments of $\log_{10} X$ with those of $\log_{10}(\sum_{i=1}^K X_i)$:

$$\int_0^\infty (\log_{10} x)\, p_X(x) dx = \int_0^\infty (\log_{10} x)\, p_{(\sum_{i=1}^K X_i)}(x) dx, \tag{2a}$$

$$\int_0^\infty (10 \log_{10} x - \mu_Y)^2 p_X(x) dx = \int_0^\infty \left(10 \log_{10} x - \mu_{\hat{Y}}\right)^2 p_{(\sum_{i=1}^K X_i)}(x) dx, \tag{2b}$$

where μ_Y and $\mu_{\hat{Y}}$ are the mean values of $Y = 10 \log_{10} X$ and $\hat{Y} = 10 \log_{10} \left(\sum_{i=1}^K X_i \right)$, respectively. While the match is exact for $K = 2$, an iterative technique needs to be used for $K > 2$. The parameters μ_Y and σ_Y are evaluated numerically. The S-Y method is more involved than the F-W method because the expectation of the logarithm sum cannot be directly written in terms of the expectations of the individual RVs. It is inaccurate near the tail portion of the pdf and can significantly underestimate small values of the CDF [10].

Interpreting the moments as weighted integrals of the pdf, both the F-W method and the S-Y method can be generalized by the following system of equations for $m = 1$ and 2:

$$\int_0^\infty w_m(x) p_X(x) dx = \int_0^\infty w_m(x) p_{(\sum_{i=1}^K X_i)}(x) dx. \tag{3}$$

The F-W method uses the weight functions $w_1(x) = x$ and $w_2(x) = (x - \mu_X)^2$, both of which monotonically increase with x. Thus, errors in the tail portion of the sum pdf are penalized more. This explains why the F-W method tracks the tail portion well. On the other hand, the S-Y method employs the weight functions $w_1(x) = \log_{10} x$ and $w_2(x) = (\log_{10} x - \mu_Y)^2$. Due to the singularity of $\log_{10} x$ at $x = 0$, mismatches near the origin are severely penalized by both these weight functions. Compared to the F-W method, the S-Y method gives less weight to the pdf tail. For these reasons, it does a better job tracking the head portion of the pdf. However, both the F-W and the S-Y methods use fixed weight functions and offer no way of overcoming their respective shortcomings.

Similarly, Schleher's cumulants matching method [8] accords a polynomially increasing penalty to the approximation error in the tail portion of the pdf. This is because the first three cumulants are, in effect, the first three central moments of an RV. By plotting the x-axis in dB scale on lognormal paper, the Beaulieu-Xie method also accords a higher priority to the tail portion.

III. Lognormal Sum Approximation Using Gauss-Hermite Expansion of MGF

A. Motivation

The moment generating function (MGF) of an RV X is defined as

$$\Psi_X(s) = \int_0^\infty \exp(-sx) p_X(x) dx, \quad (\text{Re}(s) \ge 0). \tag{4}$$

The simplicity of the F-W method arises from the fact that the mean and variance of a sum of independent RVs can be written directly as the sum of the mean and variance of the individual RVs. The MGF of the sum of independent RVs also possesses this desirable property that it can be written directly in terms of the MGFs of the individual RVs as follows:

$$\Psi_{(\sum_{i=1}^K X_i)}(s) = \prod_{i=1}^K \Psi_{X_i}(s), \quad (\text{Re}(s) \ge 0). \tag{5}$$

From (3) and (4), the MGF can also be interpreted as a weighted integral of the pdf, $p_X(x)$, with the weight function being a monotonically decreasing exponential function, $\exp(-sx)$, in x. Varying s adjusts, as required, the weights allocated to the head and tail portions of the sum pdf. Figure 1 compares, in log scale, the absolute values of the various weight functions discussed above.

Based on the discussion above, we can see that the MGF posses two desirable properties. First, the MGF is a weighted integral of the pdf with a weight function that is adjustable. Second, the MGF of the sum pdf can be easily expressed as the product of the MGFs of the individual RVs. These two properties make the MGF a preferable candidate for the lognormal sum approximation problem. We therefore propose the following method based on matching the MGF of the lognormal distribution.

B. MGF-based Lognormal Sum Approximation

While no general closed-form expression for the lognormal MGF is available, it can be readily expressed by a series expansion based on Gauss-Hermite integration.[2] We restrict the development to real values of s as it still provides considerable flexibility in adjusting the weighted integrals. The MGF of a lognormal RV X can be written as

$$\Psi_X(s) = \int_0^\infty \frac{\xi \exp(-sx)}{x\sigma_Y\sqrt{2\pi}} \exp\left[-\frac{(\xi \log_e x - \mu_Y)^2}{2\sigma_Y^2}\right] dx, \quad (6a)$$

$$= \sum_{n=1}^{N} \frac{w_n}{\sqrt{\pi}} \exp\left[-s \exp\left(\frac{\sqrt{2}\sigma_Y a_n + \mu_Y}{\xi}\right)\right] + R_N, \quad (6b)$$

where μ_Y and σ_Y are the mean and standard deviation of the Gaussian RV $Y = 10\log_{10} X$. Eqn. (6b) is the Gauss-Hermite series expansion of the MGF function, N is the Hermite integration order, $\xi = 10/\log_e 10$ is a scaling constant, and R_N is a remainder term. The weights, w_i, and the abscissas, a_i, are tabulated in [14, Tbl. 25.10] for $N \leq 20$. From (6b), we can define the Gauss-Hermite representation of the MGF, $\widehat{\Psi}_X$, by removing R_N as follows:

$$\widehat{\Psi}_X(s;\mu,\sigma) = \sum_{n=1}^{N} \frac{w_n}{\sqrt{\pi}} \exp\left[-s \exp\left(\frac{\sqrt{2}\sigma a_n + \mu}{\xi}\right)\right]. \quad (7)$$

Figure 2 shows the impact of N on the accuracy of the Gauss-Hermite representation of the MGF. We have found that the lognormal MGF, $\Psi_X(F)$, can be accurately approximated by its Gauss-Hermite expansion $\widehat{\Psi}_X(s;\mu,\sigma)$ with $N = 12$.

The lognormal sum $\sum_{i=1}^{K} X_i$ can now be approximated by a lognormal RV $X = 10^{0.1Y}$, where $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$, by matching the MGF of X with the MGF of $\sum_{i=1}^{K} X_i$ at two different values of s: s_1 and s_2. This sets up a system of two independent equations to calculate μ_Y and σ_Y^2, as follows:

$$\sum_{n=1}^{N} \frac{w_n}{\sqrt{\pi}} \exp\left[-s_m \exp\left(\frac{\sqrt{2}\sigma_Y a_n + \mu_Y}{\xi}\right)\right]$$
$$= \prod_{i=1}^{K} \widehat{\Psi}_X(s_m;\mu_{Y_i},\sigma_{Y_i}), \quad \text{for } m = 1 \text{ and } 2. \quad (8)$$

Note that the right hand side of the above two equations is a constant number. These non-linear equations in μ_Y and σ_Y can be readily solved numerically using standard functions such as `fsolve` in Matlab and `NSolve` in Mathematica.

[2] A formula for the MGF, in the form of an infinite series, was derived by Naus [15] for the special case of the sum of two independent and identically distributed lognormal RVs.

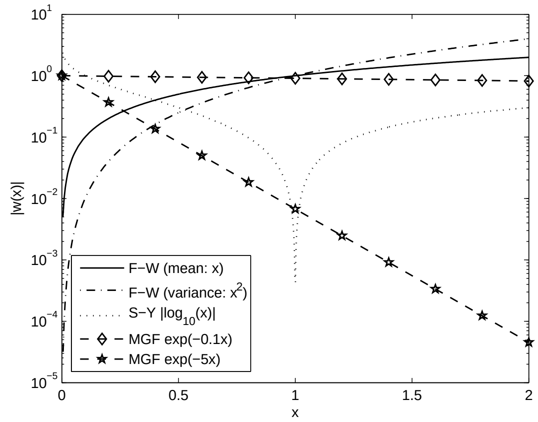

Fig. 1. Weight functions employed by F-W, S-Y, and the proposed MGF-based method

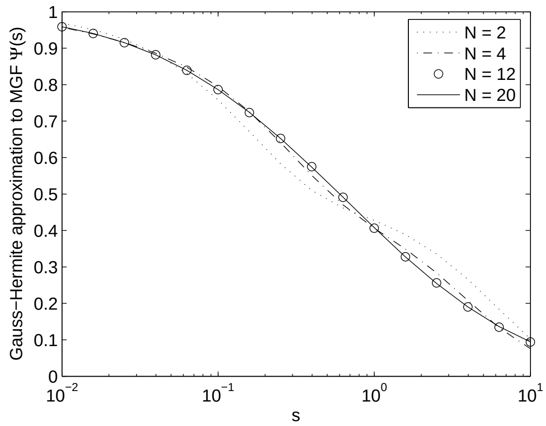

Fig. 2. $\widehat{\Psi}_X(s;\mu,\sigma)$ as a function of s for different Hermite integration orders, N ($\mu = 0$ dB and $\sigma = 8$ dB)

The values of μ_Y and σ_Y can be accurately determined using $N = 12$. We have found that even $N = 6$ is often sufficient. This is because the form of (8) makes the desired parameters insensitive to MGF approximation errors. The number of terms is small compared to the 20 to 40 terms required to achieve numerical accuracy in the S-Y method [16]. Furthermore, unlike the S-Y method, no iteration in K is required – the right hand side of (8) only needs to be computed twice (at s_1 and s_2) for any K.

Most importantly, as highlighted before, the penalty for pdf mismatch can be adjusted by choosing s appropriately. Increasing s penalizes more the errors in approximating the head portion of the sum pdf, while reducing s penalizes errors in the tail portion. For example, when the lognormal sum arises because various signal components add up and the main performance metric is the signal outage probability, the tail of the CDF needs to be computed accurately. On the other hand, the head portion of the sum pdf needs to be calculated accurately in outage probability calculations when the lognormal sum appears in the denominator term only, for example, as the sum of the powers of co-channel interferers. The proposed method can handle both of these applications by using different pairs (s_1, s_2). Guidelines for choosing (s_1, s_2) are developed in the following section.

IV. Numerical Examples

Given the importance of co-channel interference (CCI) in cellular systems, we consider the downlink of a representative hexagonal cellular layout with one and two rings of interfering base stations (BS) to compare the performance of the proposed method with other methods. Due to pathloss, the mean values of the CCI from the second-tier interferers differ considerably from those of the first-tier interferers.

Figure 3 shows the cell layout with 6 first-tier interferers and 12 second-tier interferers and the location of the mobile station (MS) under consideration. BS 0 is the serving BS. The i^{th} lognormal RV, X_i, observed by the MS is given by $X_i = \gamma_0 \left(\frac{d_i}{R}\right)^{-\eta} 10^{0.1Y_i}$, where γ_0 is the signal to noise ratio (SNR) at the corner of the center cell, R is the cell radius, η is the pathloss exponent, d_i is the distance between the i^{th} BS and the MS, and Y_i is a zero-mean Gaussian RV with variance σ. The examples that follow use $\gamma_0 = 10$ dB, $\eta = 3.7$, and assume that the MS is at a distance of $R/2$ from the serving (central) BS.[3]

In the examples, we plot the CDF and complementary CDF (CCDF) and use these results to provide guidelines on choosing robust values for s_1 and s_2 that work for a wide range of system parameters. As mentioned, small values of the CDF reveal the accuracy in tracking the head portion of the pdf, while small values of the CCDF reveal the accuracy in tracking the tail portion of the pdf.

Figure 4 plots the CDF of the CCI from the first-tier interferers, which corresponds to the sum of $K = 6$ non-identical lognormal RVs, for $\sigma = 8$ dB. The CDFs of the lognormal approximations from the proposed method, F-W method, and the S-Y method are compared with that from a Monte Carlo simulation, which generated 10^6 samples. It can be seen that the proposed method matches the head portion of the distribution function very well when $(s_1, s_2) = (0.2, 1.0)$, and is more accurate than both the F-W and the S-Y methods. The CCDF for the same parameters is plotted in Figure 5. While the S-Y method diverges from the actual CCDF in this scenario, the proposed method matches the simulation results well for $(s_1, s_2) = (0.001, 0.005)$, and is as accurate as the F-W method. The inevitable trade-off that needs to be made in approximating both the head and tail portions of the pdf implies that the same (s_1, s_2) values cannot be used to accurately match the head and the tail portions.

The effect of increasing the number of interferers is shown in Figure 6, which plots the CDF of the CCI from both first-tier and second-tier interferers, *i.e.*, $K = 18$. It can be seen from these two figures that $(s_1, s_2) = (0.2, 1.0)$ provides a good fit for various values of σ and K for approximating the head portion of the pdf. Similarly, $(s_1, s_2) = (0.001, 0.005)$ is suitable for approximating the tail of the pdf. These values of s_1 and s_2 were found to be suitable for several other system parameters, as well.

[3]The pathloss factor $\left(\frac{d_k}{R}\right)^{-\eta}$ affects only the mean of X_k, but not its variance.

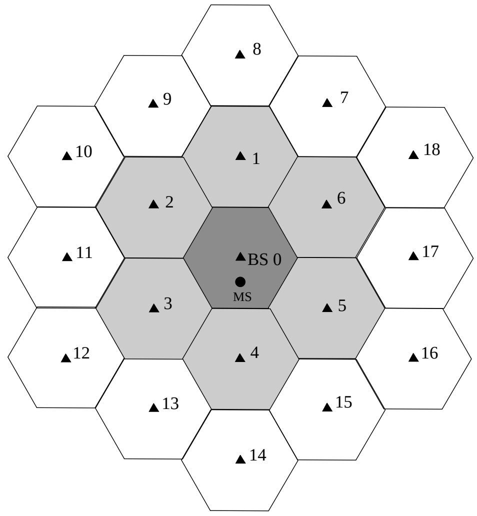

Fig. 3. Cellular layout with up to two rings of downlink co-channel interferers

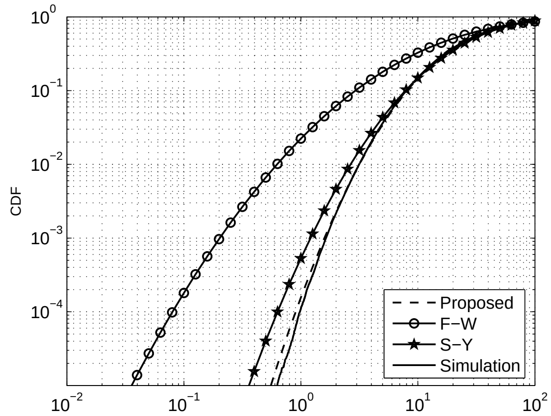

Fig. 4. CDF of co-channel interference from first-tier interferers ($K = 6$) for $\sigma = 8$ dB

Figure 7 shows an application of this method to the problem of computing the outage probability of an interference-limited cellular system with 6 first-tier co-channel interferers. The signal component and the interferers all undergo lognormal shadow fading with a variance of σ dB. An outage is declared if the SIR falls below 12 dB. The outage probabilities obtained analytically by employing the proposed method, with $(s_1, s_2) = (0.2, 1.0)$, are compared with results from the F-W and S-Y methods, and from Monte Carlos simulations, for various σ. The simulation results were averaged over 100,000 independent trials. Excellent agreements are observed between the the approximated analytical results and the simulation results.

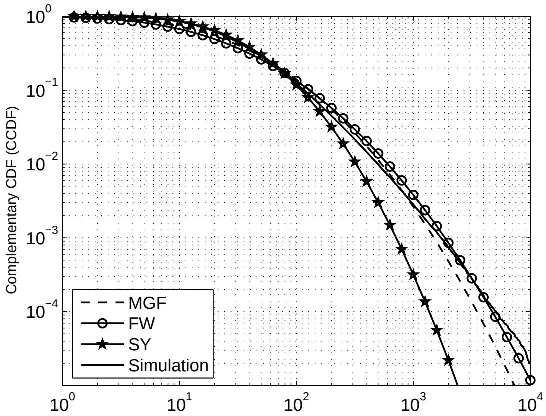

Fig. 5. CCDF of co-channel interference from first-tier interferers ($K = 6$) for $\sigma = 8$ dB

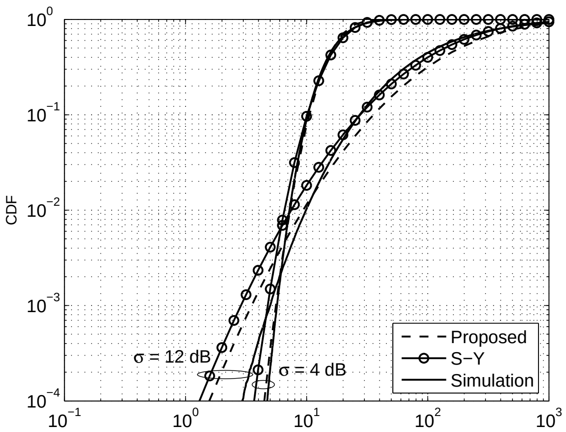

Fig. 6. CDF of co-channel interference from both first and second-tier interferers ($K = 18$) for $\sigma = 4$ dB and $\sigma = 12$ dB

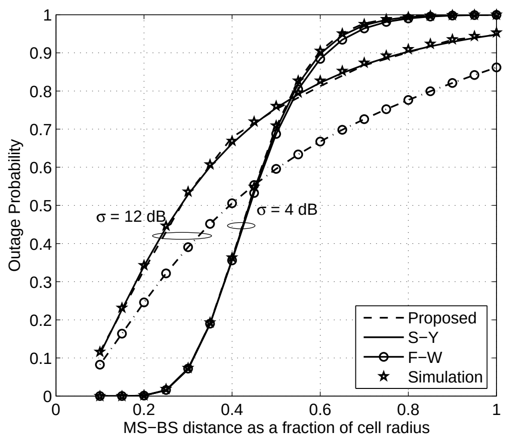

Fig. 7. Outage probability of system with first-tier interferers ($K = 6$)

V. Conclusions

We proposed a simple and novel method to approximate the sum of several independent, but not necessarily identical, lognormal random variables with a single lognormal random variable. Motivated by an interpretation of MGF as a weighted integral of the pdf, the method computes the approximating distribution parameters by matching the MGF of lognormal sum with that of the approximating RV at two real and positive points, s_1 and s_2. Matching at $s_1 = 0.2$ and $s_2 = 1$ accurately approximates the head portion of the lognormal sum pdf, while matching at $s_1 = 0.001$ and $s_2 = 0.005$ accurately approximates the tail portion of the pdf. This choice was shown to be appropriate for a wide range of system parameters.

The weighted integral interpretation also explained the observed shortcomings of some of the methods currently available in the literature. The proposed method provides the flexibility to handle the inevitable trade-off that needs to be made in approximating different regions of the pdf. Its computational complexity is similar to that of the Schwartz-Yeh method.

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