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***Abstract*—In this paper, we are concerned with the design and analysis of joint source-channel coding schemes for block fading channels with relay-assisted distributed spatial diversity. Assuming a progressive image coder with a constraint on the transmission bandwidth, we formulate a joint source-channel rate allocation scheme that maximizes the expected source throughput. Specifically, using Gaussian as well as BPSK inputs on flat Rayleigh fading channels, we lower bound the average packet error rate by the corresponding mutual information outage probability, and derive the average throughput expression as a function of channel code rates as well as channel SNR for both a frequency-division multiplexing-based baseline system without relaying, and a half-duplex relay system with a decode-and-forward protocol. At high signal-to-noise ratio (SNR), for the systems considered in this paper, we show that our rate optimization problem is a convex function of the channel code rates, and we show that a known recursive algorithm can be used to predict the performance of both systems.**

***Index Terms*—Joint source channel coding, progressive image communication, unequal error protection, cooperative relaying.**

## I. Introduction

When we transmit source bits through an unreliable channel, we need channel bits to protect information from channel noise. However, due to limited bandwidth, power or delay constraints, channel resources should be shared by source and channel bits optimally in the sense of distortion or throughput. Therefore, the problem is how to decide the ratio of source and channel bits under the system constraints. This problem has been considered for progressive source coding such as embedded zerotree wavelet (EZW) coding [1], set partitioning in hierarchical trees (SPIHT) [2] and JPEG2000 [3], where the decoding of the progressive image stops at the first packet error. Due to its progressive property, we need to push the first error event as far back as possible in the packet stream for better performance. To do that, earlier packets generally need more protection than others, and therefore unequal error protection (UEP) outperforms equal error protection (EEP).

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Many studies have appeared on the design and analysis of joint source-channel coding schemes for various channels of both theoretical and practical interest. For example, in [4], the progressive transmission of images jointly with rate compatible punctured convolutional (RCPC) codes was proposed and investigated for binary symmetric channels. Later, this was extended to fading channels with a product code structure [5], where the product codes consist of RCPC codes in a row and Reed Solomon (RS) codes in a column. In [6], optimization using dynamic programming was proposed to choose block length as well as appropriate code rate. To do that, the authors defined a general performance measure, where the performance measure could be mean-squared error (MSE), peak signal-to-noise ratio (PSNR), or the number of received source bits. A more computationally efficient algorithm was proposed by Chande and Farvardin in [7], where the optimal UEP solution was computed recursively. Later, Stanković et al. introduced a faster algorithm to find the rate-optimal UEP solution by computing the run-length profile of the code rates [8]. Nosratinia et al. proposed a parametric approach for source-channel bit allocation, where the exact bit error rate (BER) corresponding to each coding rate was modelled empirically for certain channels [9]. An information-theoretic approach for joint source-channel bit allocation was also presented by Gunduz et al. [10] and Etemadi et al. [11], where both considered layered transmission of successively refinable Gaussian sources over quasi-static fading channels, and found theoretical bounds for expected distortion using well-known rate-distortion expressions for a Gaussian source. Recently, joint source-channel coding was considered in cooperative relaying systems, where cooperative relaying can improve channel performance by providing additional diversity. Gunduz et al. [12] considered joint source-channel coding for cooperative relaying systems when channel state information is only known at the receiver (CSIR) and quasi-static channels are assumed, and investigated the performance in terms of the distortion exponent. Shutoy et al. [13] proposed layered-cooperative coding to exploit a relay's diversity gain for a video source. Finally, in [14], hybrid error protection was proposed, where cooperative diversity was considered as an additional protection tool beyond forward error correction.

In this paper, we use an information-theoretic framework to analyze the source-channel rate allocation to maximize the throughput of a progressive image in a system either with or without a cooperative relay. This analysis provides an approximate bound on the system performance at high SNR in terms of average throughput, which may not be optimal in the sense of distortion. However, the throughput-optimal approach has some advantages over the distortion-optimal approach. First, the throughput-optimal approach can be analyzed mathematically without considering the source characteristics. And, the UEP solution can be found at the transmitter and receiver independently without any additional signaling between them, since the average channel gain, which is needed in the throughput-optimal approach, is known by both the transmitter and the receiver. Moreover, the throughput-optimal approach can allow one to find the distortion-optimal solution using a local search algorithm and the source statistics in linear time [28]. We study both Gaussian and symmetric BPSK inputs over block fading channels, and get the mutual information (MI) outage probability, which is a lower bound on the actual packet error rate (PER) of a BPSK-based system [15],[16]. Although there are well-known expressions for mutual information and outage probability for Gaussian inputs, a closed-form expression for the capacity when either PSK or QAM inputs is used is not known. However, both bounds and approximations have been widely studied (e.g.,[17]-[20]). In this paper, we use [18] to find an approximation for the MI outage probability for BPSK inputs. Then, we derive an average source throughput expression using MI outage probability and the progressive property of the source. We prove that this rate-optimization problem is a convex function of the channel code rates at high SNR, and the solution can be found by using a recursive algorithm introduced in [8]. Our predicted system performance and average channel code rates upper bound the simulation results.

The rest of this paper is organized as follows. In Section II, we introduce the system model for the transmission of progressive images. Using both Gaussian and BPSK inputs, in Section III, we present expressions for the average source throughput for the baseline as well as the relay-based systems. In Section IV, we provide a simple algorithm to determine the optimal channel code rates under a bandwidth constraint. Numerical and simulation results are presented in Section V, and we conclude this work in Section VI.

## II. System Model

We consider a general progressive image transmission system consisting of a single source and destination either with or without relays. The image source rate is denoted by $\mathcal{R}_s$ pixels/sec. If we assume $r_s$ bits-per-pixel (bpp), then the total source rate is $\mathcal{R}_s r_s$ bits/sec. The source bit stream is packetized into $M$ packets, which are protected by suitable channel codes. All the packets are assumed to have equal channel codeword length, $n$. The total bandwidth available for the source is $W$ Hz, whereas the duration of a code symbol is denoted by $T_s$. Assuming a Nyquist pulse-shaping filter at the transmitter, we have

$$W = \frac{1+\beta}{T_s}, \tag{1}$$

where $\beta$ is the roll-off factor of the pulse-shaping filter.

We assume the channels between all the nodes are random, independent, and constant over the packet duration. The coherence bandwidth is assumed to be the same as the bandwidth of a sub-channel, where the bandwidth of a sub-channel is defined as $W/N$ and $N$ is the number of relays. We assume Rayleigh fading channel, so the square of the fading amplitude follows an exponential distribution, and the additive noise is i.i.d. white Gaussian and independent for each channel. We assume that the receivers of the relays and the destination know the instantaneous channel realizations, and the transmitters of the source and the relays know only the mean channel gain.

The destination combines the received channel code symbols on the $N$ sub-channels using maximal ratio combining (MRC). We assume that the system is able to detect any errors and halts decoding of subsequent packets following an erroneous packet.

### A. Baseline System

In the baseline system, the transmission bandwidth $W$ is equally divided into $N$ uncorrelated sub-bands and the source repeats each packet over the $N$ sub-channels. The bit duration of each sub-channel is $T_{s_1} = NT_s$. If $K_j$ denotes the number of information bits in the $j$th packet, where $j = 1, \ldots, M$, then the channel code rate of the $j$th packet is $r_c^{\mathrm{BL}}(j) = K_j/n$, where $n$ is the packet length in bits. The total transmission time for a given image is $M \times n \times T_{s_1}$. In this time, the total number of source bits generated is $\mathcal{R}_s r_s M n T_{s_1}$ which is assumed to be equal to $\sum_{j=1}^{M} K_j = n \sum_{j=1}^{M} r_c^{\mathrm{BL}}(j)$. That is, we have the rate constraint

$$n \sum_{j=1}^{M} r_c^{\mathrm{BL}}(j) = \mathcal{R}_s r_s M n T_{s_1} = N \mathcal{R}_s r_s M n \frac{1+\beta}{W} \leq Mn. \tag{2}$$

If $T_b^{\mathrm{BL}}(j)$ denotes the information bit duration, and $R_b^{\mathrm{BL}}(j) = 1/T_b^{\mathrm{BL}}(j)$ denotes the corresponding bit rate for the $j$th packet, then, using the relation $nT_{s_1} = K_j T_b^{\mathrm{BL}}(j)$, we have

$$R_b^{\mathrm{BL}}(j) = \frac{K_j}{n}\frac{1}{T_{s_1}} = \frac{r_c^{\mathrm{BL}}(j)}{NT_s} = \tilde{r}_c^{\mathrm{BL}}(j)\frac{W}{N}, \tag{3}$$

where $\tilde{r}_c^{\mathrm{BL}}(j) = r_c^{\mathrm{BL}}(j)/(1+\beta)$.

Let $\alpha_{j,l}$ denote the channel fading amplitude experienced by the $j$th packet on the $l$th sub-channel. We assume that the $\alpha_{j,l}$, for $j = 1, \ldots, M$ and $l = 1, \ldots, N$, are independent and identically distributed (i.i.d) Rayleigh random variables, with second moment $E[\alpha_{j,l}^2] = \Omega_1$. The total average transmission power budget at the source is $P_T$. Therefore, the average power per sub-channel is $P_1 = P_T/N$.

### B. Half-duplex Relay System

We assume $N$ relay nodes, each of them and the source occupying a bandwidth of $W/N$, as presented in Fig. 1. The source transmits each packet sequentially through one of $N$ sub-channels. Each packet is received by the destination as well as by all the $N$ relays. The packets are assumed to be further protected by suitable cyclic redundancy check (CRC) codes to aid verification of their integrity. With this, only those

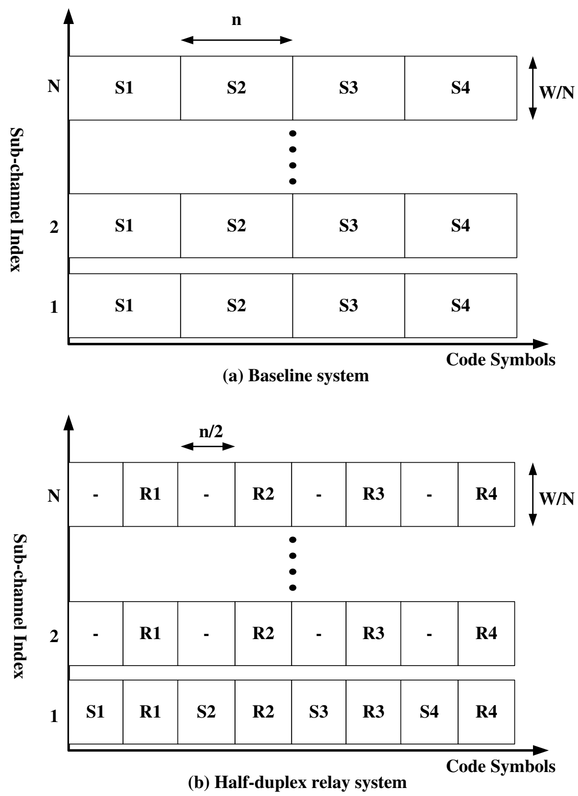


Fig. 1. (a) Baseline system and (b) half-duplex relay system, where S$i$ and R$i$ are the $i$th packet transmission at the source and the relay, respectively.

relay nodes that successfully decode the source packets re-encode and forward them to the destination. For simplicity, we assume that the probability of undetected error is very low, and can be ignored. The destination combines all the packets that it receives from the source and the relay nodes in an appropriate manner before proceeding with channel decoding. For the $l$th packet, we denote by $\alpha_{sd,l}$ the fading amplitude on the path from the source to the destination, $\alpha^j_{sr,l}$ the fading amplitude on the path from the source to the $j$th relay node, and $\alpha^j_{rd,l}$ the fading amplitude on the path from the $j$th relay node to the destination. Similar to the baseline system, we assume that $\alpha_{sd,l}$, $\alpha^j_{sr,l}$ and $\alpha^j_{rd,l}$, for $j = 1, \ldots, N$ and $l = 1, \ldots, M$, are independent and Rayleigh distributed, with second moments $E[\alpha^2_{sd,l}] = \Omega_{sd}$, $E[(\alpha^j_{sr,l})^2] = \Omega^j_{sr}$ and $E[(\alpha^j_{rd,l})^2] = \Omega^j_{rd}$.

If we denote by $L_j$ the number of information bits in the $j$th packet, then the code rate of the $j$th packet with cooperation is $r_c^{\mathrm{CoOp}}(j) = L_j/n$, $j = 1, \ldots, M$. In the half duplex relay system, where the relays transmit and receive in different time slots, $2M$ time slots are required to transmit $M$ packets. Since the bit duration of the half duplex relay system, $T_{s_2}$, is equal to that of the baseline system, the length of each time slot is $(n/2)T_{s_2} = (n/2)NT_s$, which is half of a packet transmission time. This leads to the following rate constraint with cooperation :

$$\sum_{j=1}^{M} L_j = \mathcal{R}_s r_s M T_{s_2} n/2 \leq Mn/2$$
$$\Rightarrow \frac{1}{M}\sum_{j=1}^{M} r_c^{\mathrm{CoOp}}(j) = \frac{1}{2} N \mathcal{R}_s r_s T_s \leq \frac{1}{2}, \tag{4}$$

where the effective channel code rate of the $j$th packet, $r_c^{\mathrm{CoOp}}(j) = L_j/n$. If we denote by $P_s$ the average transmit power of the source, and by $P_j$ the average transmit power of the $j$th relay node, then we have the following energy constraint :

$$P_s(n/2)T_{s2} + \sum_{j=1}^{N} P_j (n/2) T_{s2} = P_T n T_{s1}$$
$$\Rightarrow P_s + \sum_{j=1}^{N} P_j = 2P_T, \tag{5}$$

where $P_T$ is the average transmission power of the source without cooperation.

## III. Throughput Performance Analysis

In this section, by assuming large block lengths and block fading of the channel over the packet duration $nT_{s_1}$, we lower bound the actual PER by the channel mutual information (MI) outage probability [22], which is defined [23] as the probability that the instantaneous MI observed by a packet, which is a random variable (r.v.), is below the attempted information bit rate. Then, we derive the average throughput expression for each system using the MI outage probability.

### *A. Gaussian Inputs*

*1) Baseline System:* Let $\pi_j^{\mathrm{BL}}$ denote the MI outage probability for the $j$th packet, which is a lower bound of the PER, at the destination. For the $j$th packet, the MI (in bits/sec) is

$$\mathrm{MI}^{\mathrm{BL}}(j) = \frac{W}{N}\log_2\left(1 + \sum_{l=1}^{N} \frac{P_1}{N_0 \frac{W}{N}} \alpha^2_{j,l}\right). \tag{6}$$

The corresponding MI outage probability is

$$\begin{aligned}
\pi_j^{\mathrm{BL}} &= \mathrm{Prob}\left(\mathrm{MI}^{\mathrm{BL}}(j) < R^{\mathrm{BL}}_{b,j}\right) \\
&= \mathrm{Prob}\left(\frac{W}{N}\log_2\left(1 + \sum_{l=1}^{N}\frac{P_1}{N_0\frac{W}{N}}\alpha^2_{j,l}\right) \leq \frac{W\tilde{r}_c^{\mathrm{BL}}(j)}{N}\right) \\
&= \mathrm{Prob}\left(\sum_{l=1}^{N}\frac{\alpha^2_{j,l}}{\Omega_1} \leq \frac{2^{\tilde{r}_c^{\mathrm{BL}}(j)} - 1}{\overline{\Gamma}}\right) \\
&= \gamma_{\mathrm{inc}}\left(\frac{2^{\tilde{r}_c^{\mathrm{BL}}(j)} - 1}{\overline{\Gamma}}, N\right),
\end{aligned} \tag{7}$$

where we have used the fact that $\zeta \triangleq \sum_{l=1}^{N} \alpha^2_{j,l}/\Omega_1$ is a Gamma-distributed r.v. with probability density function (PDF)

$$f_\zeta(x) = \frac{e^{-x}x^{N-1}}{\Gamma(N)} \quad x \geq 0, \tag{8}$$

and where $\Gamma(n) = \int\limits_{u=0}^{\infty} e^{-u}u^{n-1}du$ is the standard Gamma function [26], and $\gamma_{\text{inc}}(x,n)$ in (7) is the incomplete Gamma function, which is defined as [26]

$$\gamma_{\text{inc}}(x,n) \triangleq \frac{1}{\Gamma(n)} \int\limits_{u=0}^{x} e^{-u}u^{n-1}du. \tag{9}$$

In (7) $\overline{\Gamma} = P_1\Omega_1 N/(N_0 W) = P_T\Omega_1/(N_0 W)$ is the average received signal-to-noise ratio (SNR) per sub-channel.

Due to the progressive nature of the source, only the packets decoded successfully until the first decoding failure are used by the source decoder for reconstructing the source. Let us denote by $\mathsf{S}_k^{\text{BL}}$ the probability of successfully receiving $k$ source packets at the input of the source decoder. Then, we have

$$\begin{aligned}
\mathsf{S}_0^{\text{BL}} &= \pi_1^{\text{BL}} \\
\mathsf{S}_k^{\text{BL}} &= \pi_{k+1}^{\text{BL}} \prod_{i=1}^{k}(1-\pi_i^{\text{BL}}) \quad k = 1,\ldots,M-1 \\
\mathsf{S}_M^{\text{BL}} &= \prod_{i=1}^{M}(1-\pi_i^{\text{BL}}).
\end{aligned} \tag{10}$$

The average number of source bits successfully decoded by the source decoder in the baseline system is

$$\begin{aligned}
\overline{\mathcal{T}}_{\text{BL}}(\underline{r}_c^{\text{BL}}) &= \sum_{j=1}^{M}\left(\sum_{i=1}^{j} K_i\right)\mathsf{S}_j^{\text{BL}} \\
&= n(1+\beta)\sum_{j=1}^{M}\left(\sum_{i=1}^{j}\tilde{r}_c^{\text{BL}}(i)\right)\mathsf{S}_j^{\text{BL}},
\end{aligned} \tag{11}$$

where $\underline{r}_c^{\text{BL}} = \left(\tilde{r}_c^{\text{BL}}(1),\ldots,\tilde{r}_c^{\text{BL}}(M)\right)$. The source-channel rate allocation problem for the baseline system can then be stated as

$$\text{maximize} \quad \overline{\mathcal{T}}_{\text{BL}}(\underline{r}_c^{\text{BL}}) \tag{12}$$

$$\text{subject to} \quad \sum_{j=1}^{M}\tilde{r}_c^{\text{BL}}(j) \le \frac{M}{1+\beta}. \tag{13}$$

*2) Half-duplex Relay System:* We denote by $q(i,j)$ the probability of decoding error of the $j$th packet at the $i$th relay node, which is computed as follows: Let $\text{MI}(i,j)$ denote the MI of the $j$th packet at the $i$th relay, then

$$\begin{aligned}
\text{MI}(i,j) &= \frac{W}{N}\log_2\left(1 + \frac{P_s}{N_0\frac{W}{N}}(\alpha_{sr,j}^i)^2\right) \\
&= \frac{W}{N}\log_2\left(1 + \overline{\gamma}_{sr}^i N \frac{\left(\alpha_{sr,j}^i\right)^2}{\Omega_{sr}^i}\right),
\end{aligned} \tag{14}$$

where $\overline{\gamma}_{sr}^i = \frac{P_s}{N_0 W}\Omega_{sr}^i$, $i = 1,\ldots,N$. The transmission rate of the $j$th packet with cooperation is $R_b^{\text{CoOp}}(j) = 2\tilde{r}_c^{\text{CoOp}}(j)W/N$, where $\tilde{r}_c^{\text{CoOp}}(j) = r_c^{\text{CoOp}}(j)/(1+\beta)$. Then,

$$\begin{aligned}
q(i,j) &= \text{Prob}\left(\text{MI}(i,j) < R_b^{\text{CoOp}}(j)\right) \\
&= \text{Prob}\left(\frac{\left(\alpha_{sr,j}^i\right)^2}{\Omega_{sr}^i} < \frac{2^{2\tilde{r}_c^{\text{CoOp}}(j)}-1}{\overline{\gamma}_{sr}^i N}\right) \\
&= \gamma_{\text{inc}}\left(\frac{2^{2\tilde{r}_c^{\text{CoOp}}(j)}-1}{\overline{\gamma}_{sr}^i N}, 1\right),
\end{aligned} \tag{15}$$

which follows from the fact that $(\alpha_{sr,j}^i)^2/\Omega_{sr}^i$ is exponentially-distributed with unit mean.

Let $\mathcal{D}_j$ denote the set of relay nodes that successfully decode the $j$th source packet. Then the probability that relays in the set $\mathcal{D}_j$ are only able to decode $j$th source packet is

$$\text{Prob}(\mathcal{D}_j) = \left\{\prod_{i\in\mathcal{D}_j}(1-q(i,j))\right\} \times \left\{\prod_{k\notin\mathcal{D}_j} q(k,j)\right\} \tag{16}$$

owing to the independence of the decoding errors at the relays, which is attributed to the spatial independence of the channel fading.

At the destination, we assume that each source packet is maximal-ratio combined. Then, the MI of the $j$th packet at the destination, conditioned on $\mathcal{D}_j$, is

$$\begin{aligned}
\text{MI}(\mathcal{D}_j) &= \frac{W}{N}\log_2\left(1 + \frac{P_s}{N_0\frac{W}{N}}\alpha_{sd,j}^2 + \right. \\
&\qquad \left. \sum_{l\in\mathcal{D}_j}\frac{P_l}{N_0\frac{W}{N}}(\alpha_{rd,j}^l)^2\right) \\
&= \frac{W}{N}\log_2\left(1 + N\overline{\gamma}_{sd}(\alpha_{sd,j})^2/\Omega_{sd} + \right. \\
&\qquad \left. N\sum_{l\in\mathcal{D}_j}\overline{\gamma}_{rd}^l(\alpha_{rd,j}^l)^2/\Omega_{rd}^l\right),
\end{aligned} \tag{17}$$

where $\overline{\gamma}_{rd}^l = \frac{P_l}{N_0 W}\Omega_{rd}^l$, $l = 1,\ldots,N$. Conditioned on $\mathcal{D}_j$, the probability of $j$th packet error at the destination is

$$\begin{aligned}
\pi_j^{\text{CoOp}}(\mathcal{D}_j) &= \text{Prob}\left(\text{MI}(\mathcal{D}_j) < \frac{2\tilde{r}_c^{\text{CoOp}}(j)W}{N}\right) \\
&= \text{Prob}\left(\overline{\gamma}_{sd}(\alpha_{sd,j})^2/\Omega_{sd} + \right. \\
&\qquad \left. \sum_{l\in\mathcal{D}_j}\overline{\gamma}_{rd}^l(\alpha_{rd,j}^l)^2/\Omega_{rd}^l < \frac{2^{2\tilde{r}_c^{\text{CoOp}}(j)}-1}{N}\right).
\end{aligned} \tag{18}$$

To simplify (18) further, define

$$Z(\mathcal{D}_j) \triangleq \overline{\gamma}_{sd}(\alpha_{sd,j})^2/\Omega_{sd} + \sum_{l\in\mathcal{D}_j}\overline{\gamma}_{rd}^l(\alpha_{rd,j}^l)^2/\Omega_{rd}^l. \tag{19}$$

Assuming $\overline{\gamma}_{sd}$, $\overline{\gamma}_{rd}^l$, $\forall\, l \in \mathcal{D}_j$, are all distinct, and using the moment generating function approach, it is easy to show that the PDF of $Z(\mathcal{D}_j)$ is given by

$$f_{Z(\mathcal{D}_j)}(z) = \lambda_0(\mathcal{D}_j)\frac{1}{\overline{\gamma}_{sd}}\exp\left(-\frac{z}{\overline{\gamma}_{sd}}\right) + \sum_{l\in\mathcal{D}_j}\lambda_l(\mathcal{D}_j)\frac{1}{\overline{\gamma}^l_{rd}}\exp\left(-\frac{z}{\overline{\gamma}^l_{rd}}\right), \quad (20)$$

where

$$\lambda_0(\mathcal{D}_j) = \prod_{l\in\mathcal{D}_j}\frac{\overline{\gamma}_{sd}}{\overline{\gamma}_{sd}-\overline{\gamma}^l_{rd}} \quad (21)$$

$$\text{and}\quad \lambda_l(\mathcal{D}_j) = \frac{\overline{\gamma}^l_{rd}}{\overline{\gamma}^l_{rd}-\overline{\gamma}_{sd}} \times \prod_{m\in\mathcal{D}_j, m\neq l}\frac{\overline{\gamma}^l_{rd}}{\overline{\gamma}^l_{rd}-\overline{\gamma}^m_{rd}}. \quad (22)$$

Upon using (20) in (18), we arrive at

$$\pi_j^{\mathrm{CoOp}}(\mathcal{D}_j) = \lambda_0(\mathcal{D}_j)\gamma_{\mathrm{inc}}\left(\frac{2^{2\tilde{r}_c^{\mathrm{CoOp}}(j)}-1}{\overline{\gamma}_{sd}N},1\right) + \sum_{l\in\mathcal{D}_j}\lambda_l(\mathcal{D}_j)\gamma_{\mathrm{inc}}\left(\frac{2^{2\tilde{r}_c^{\mathrm{CoOp}}(j)}-1}{\overline{\gamma}^l_{rd}N},1\right). \quad (23)$$

Finally, upon averaging (23) over all possible decoding sets, the average probability of error for the $j$th packet is

$$\pi_j^{\mathrm{CoOp}} = \sum_{\mathcal{D}_j}\mathrm{Prob}(\mathcal{D}_j)\pi_j^{\mathrm{CoOp}}(\mathcal{D}_j). \quad (24)$$

Once $\pi_j^{\mathrm{CoOp}}$ is found, the probability of successfully receiving $k$ packets at the input of the source decoder with cooperation, $\mathsf{P}_k^{\mathrm{CoOp}}$, can be expressed in a form similar to (10), where the $\pi_j^{\mathrm{BL}}$ in (10) is replaced by $\pi_j^{\mathrm{CoOp}}$ of (24).

The average number of source bits successfully decoded by the source decoder with cooperation is

$$\overline{\mathcal{T}}_{\mathrm{CoOp}}(\underline{r}_c^{\mathrm{CoOp}}) = \sum_{j=1}^{M}\left(\sum_{i=1}^{j}L_i\right)\mathsf{P}_j^{\mathrm{CoOp}} = n(1+\beta)\sum_{j=1}^{M}\left(\sum_{i=1}^{j}\tilde{r}_c^{\mathrm{CoOp}}(i)\right)\mathsf{P}_j^{\mathrm{CoOp}}, \quad (25)$$

where $\underline{r}_c^{\mathrm{CoOp}} = \left(\tilde{r}_c^{\mathrm{CoOp}}(1),\ldots,\tilde{r}_c^{\mathrm{CoOp}}(M)\right)$. The source-channel rate allocation problem with half-duplex relayed transmission can then be stated as follows :

$$\text{maximize}\quad \overline{\mathcal{T}}_{\mathrm{CoOp}}(\underline{r}_c^{\mathrm{CoOp}}) \quad (26)$$

$$\text{subject to}\quad \sum_{j=1}^{M}\tilde{r}_c^{\mathrm{CoOp}}(j) \leq \frac{M}{2(1+\beta)}. \quad (27)$$

### *B. BPSK Inputs*

In order to apply the results of an information-theoretic analysis to practical system design, BPSK inputs are considered. Although a closed-form expression for the symmetric capacity with respect to BPSK inputs is not known, it is possible to compute the symmetric capacity numerically. However, in this paper, we use both bounds and an approximation for symmetric capacity instead of numerical computation. Both upper and lower bounds for the symmetric capacity were derived by Baccarelli in [17], and approximations were proposed in [18], [19], and [20]. In this paper, the upper bound of [17] and the approximation in [18] of the symmetric capacity for BPSK inputs are used to find an upper bound on the average throughput of the rate-optimal UEP for the system. From (10) in [17], an upper bound for the MI of the baseline system is

$$\mathrm{MI}_{BPSK}^{\mathrm{BL}}(j) < \frac{W}{N}\left(1-\log_2\left(1+\exp\left(-2\sum_{l=1}^{N}\frac{P_1}{N_0\frac{W}{N}}\alpha_{j,l}^2\right)\right)\right), \quad (28)$$

and the upper bounds of MIs for the relay system are

$$\mathrm{MI}_{BPSK}(i,j) < \frac{W}{N}\left(1-\log_2\left(1+\exp\left(-2\frac{P_s}{N_0\frac{W}{N}}(\alpha_{sr,j}^i)^2\right)\right)\right), \quad (29)$$

and

$$\mathrm{MI}_{BPSK}(\mathcal{D}_j) < \frac{W}{N}\left(1-\log_2\left(1+\exp\Big(-2\frac{P_s}{N_0\frac{W}{N}}\alpha_{sd,j}^2 - 2\sum_{l\in\mathcal{D}_j}\frac{P_l}{N_0\frac{W}{N}}(\alpha_{rd,j}^l)^2\Big)\right)\right). \quad (30)$$

From (9) in [18], the approximation of the MI for the baseline system is

$$\mathrm{MI}_{BPSK}^{\mathrm{BL}}(j) \approx \frac{W}{N}\left(1-\exp\left(-2b\sum_{l=1}^{N}\frac{P_1}{N_0\frac{W}{N}}\alpha_{j,l}^2\right)\right), \quad (31)$$

and the approximations of MIs for the relay system are

$$\mathrm{MI}_{BPSK}(i,j) \approx \frac{W}{N}\left(1-\exp\left(-2b\frac{P_s}{N_0\frac{W}{N}}(\alpha_{sr,j}^i)^2\right)\right), \quad (32)$$

and

$$\mathrm{MI}_{BPSK}(\mathcal{D}_j) \approx \frac{W}{N}\left(1-\exp\left(-2b\frac{P_s}{N_0\frac{W}{N}}\alpha_{sd,j}^2 - 2b\sum_{l\in\mathcal{D}_j}\frac{P_l}{N_0\frac{W}{N}}(\alpha_{rd,j}^l)^2\right)\right), \quad (33)$$

where $b$ is a parameter that equals 0.6573 for BPSK inputs [18]. The approximated average throughput expressions of each system with respect to BPSK inputs can be derived in the same manner as those with Gaussian inputs, so details are not presented here.

## IV. Optimum Source-channel Rate Allocation

In this section, we discuss an algorithm to find the optimal source-channel rate allocation to maximize the average throughput of each system. In [8], a recursive algorithm to find the throughput-optimal solution was proposed. We show that the same algorithm can find the throughput-optimal solution for our information-theoretic framework. All objective functions for baseline as well as relay systems are concave over $\underline{r}_c^{\mathrm{BL}}$ or $\underline{r}_c^{\mathrm{CoOp}}$ at high SNR, which can be proved by showing that the Hessian matrix of the objective function is negative definite. Detailed concavity proof is presented in the appendices. Then, the optimal UEP can be found by $M$ partial differentiations as follows :

$$\frac{\partial \overline{\mathcal{T}}_M((\underline{r}_c))}{\partial r_c(i)} = 0 \quad i = 1, \ldots, M. \tag{34}$$

If there is no solution for the $i$th equation, then the $i$th element of the optimal UEP is $\min\{R\}$ or $\max\{R\}$, where $R$ is the set of possible code rates. In particular,

$$\frac{\partial \overline{\mathcal{T}}_M((\underline{r}_c))}{\partial r_c(M)} = (1 - \pi_M) - r_c(M)\frac{\partial \pi_M}{\partial r_c(M)} = 0, \tag{35}$$

where $\pi_M$ is the MI outage probability of the $M$th packet, presented in (7) and (23). That is, we can find the optimum rate of the last packet first, and then find the rate of the $(M-1)$th packet recursively. Therefore, we can find the rate-optimal UEP policy, $\underline{r}_c^* = (r_c^*(1), r_c^*(2), \cdots, r_c^*(M))$, using the algorithm of [8]:

1) Set $i = 1$ and $r_c^*(M) = \arg\max_{r_c(M)\in\mathcal{R}} \overline{\mathcal{T}}_1((r_c(M)))$ for all $r_c(M) \in (0, 1]$
2) If $i = M$, then $\underline{r}_c^* = (r_c^*(1), r_c^*(2), \cdots, r_c^*(M))$ and stop.
3) Set $i = i + 1$ and $r_c^*(M - i + 1) = \arg\max_{r_c(M-i+1)\in(0, r_c(M-i)]} \overline{\mathcal{T}}_i((r_c(M - i + 1), r_c^*(M-i), \cdots, r_c^*(M)))$. Go to step 2),

where $\overline{\mathcal{T}}_i((\underline{r}_c))$ is the average throughput expression of the last $i$ packets. Note that, at low SNR, this algorithm will find a sub-optimal UEP policy since the convexity of the average throughput was not proved.

## V. A Practical System Design Example and Simulation Results

In this section, we consider a BPSK-based practical embedded image transmission system with and without cooperative relays and then present the results of joint source-channel rate allocation based on the analysis as well as simulations. Since we find the information-theoretic rate-optimal solution, any embedded source coders and channel coding schemes can be considered. If the PER of a specific channel coding scheme is known, the rate-optimal UEP policy can be found [7] [8]. Instead, the MI outage probability can provide an approximate bound on the average throughput for high SNR, since the PER is lower bounded by MI outage probability.

### A. Simulation Setup

For the simulation, we use SPIHT to encode Lena image with 0.4 bit per pixel (bpp) of source rate. For channel encoding, RCPC and RCPT codes are considered. The RCPC codes have constraint length 3 and generator polynomial (23, 35) in octal. The rate of the mother code is 1/4, and the length of the codeword is fixed to 1000 bits, which is also the length of a packet, $L$. We consider 13 punctured code rates, $R_{RCPC}$={8/32, 8/30, 8/28, 8/26, 8/24, 8/22, 8/20, 8/18, 8/16, 8/14, 8/12, 8/10, 8/9}. Since we assumed fixed packet length, there are 13 different source block sizes corresponding to code rates. For RCPT codes, the encoder consists of two identical recursive systematic convolutional (RSC) coders with memory of 2 and generator polynomial (7,5) in octal. The rate of the mother code is 1/3 and the codeword length is 1000 bits. We also consider 9 punctured code rates, $R_{RCPT}$={8/24, 8/22, 8/20, 8/18, 8/16, 8/14, 8/12, 8/10, 8/9}. We use soft output Viterbi decoding with 10 iterations. Regarding the bit budget, we assume the total number of packets, $M$, is fixed at 100, so the optimization must find the code rates of 100 different packets which maximize average throughput. The channel is assumed to be block fading, so the channel gain is constant within a packet, but independent packet by packet. For a comparison of two systems, we assume they have same bandwidth, transmission time and energy. In the baseline system, the total transmit power is equally allocated to the subchannel, but in the half-duplex relay system, twice the total transmit power of the baseline system is equally allocated to the source as well as to the relay to make them have the same transmit energy, as shown in (5). For the half duplex relay system, we assume a single relay is located halfway between the source and destination, and the path loss exponent is set to 4.

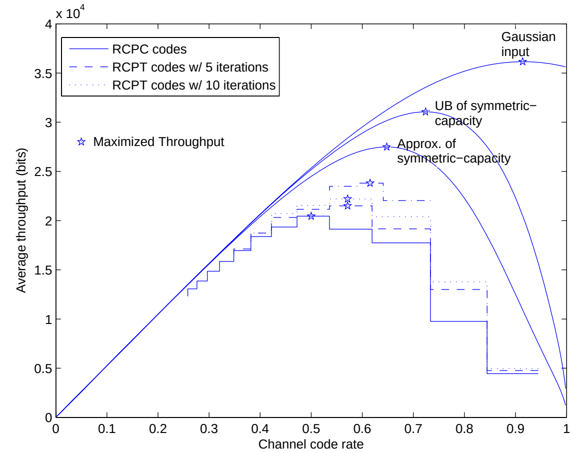


Fig. 2. EEP channel code rate profile of the baseline system at SNR of 4 dB, where three independent sub-channels are assumed.

### B. Simulation Results

In this subsection, we provide simulation results and compare them with the analytical results. Although any real number between 0 and 1 can be a code rate theoretically, we restrict the possible code rate for the analysis to be bounded by the minimum and maximum code rates of the RCPC or RCPT codes in order to get comparable results. In Fig. 2, the average throughputs of the rate-optimal EEP for the baseline system are presented, where RCPC and RCPT codes are considered. As explained in [29], the iterative decoding performance of the RCPT code approaches the maximum-likelihood (ML) bound as the number of iterations increases. Therefore, Fig. 2 shows that the rate-optimal EEP channel code rate approaches the

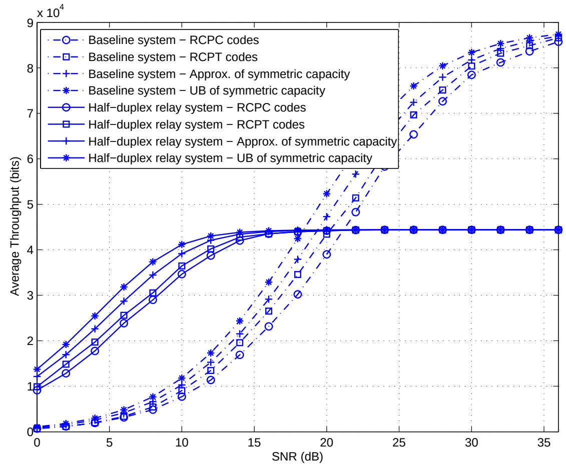


Fig. 3. Average Throughput of baseline system and half duplex relay system.

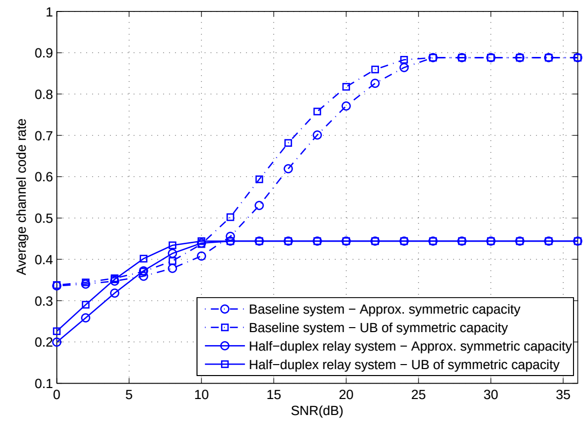


Fig. 5. Average channel code rate of baseline system and half duplex relay system. Effective code rate is presented, where effective code rate is defined as the ratio of the number of information bits and the length of packet.

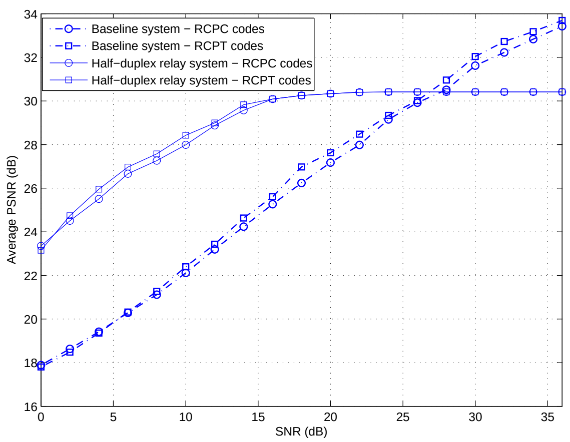


Fig. 4. Average PSNR of baseline system and half duplex relay system.

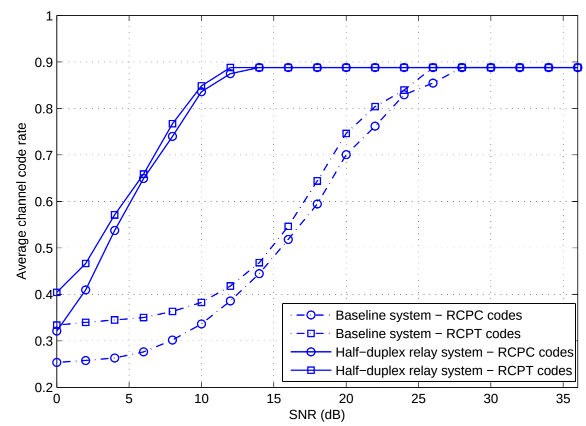


Fig. 6. Average RCPC and RCPT channel code rate of baseline system and half duplex relay system. For the half duplex relay system, code rate per time slot is presented.

analytical bound as better channel coding schemes are used. In the figure, the three solid curves represent the analytically-derived average throughputs, which are computed by using the MI outage probability of the Gaussian and BPSK inputs. For Gaussian inputs, the rate-optimal EEP code rate is approximately 0.9. However, for BPSK inputs, the rate is reduced to 0.65. On the other hand, the rate-optimal EEP channel code rate for the RCPC codes is 8/16 and the channel code rate for the RCPT codes is 8/14 with 5 iterations, 8/14 with 10 iterations, and 8/13 with 100 iterations, so the optimal code rate is approaching 0.65 asymptotically. The average throughputs and PSNRs of rate-optimal UEP policies for the baseline and the half-duplex relay systems are presented in Fig. 3 and 4. Due to the path loss reduction, as well as the cooperative diversity gain, the half duplex relay system is better than the baseline system, especially at SNR around 12 dB. However, the average throughput of the half-duplex relay system saturates at intermediate SNRs. In contrast, the throughput of the baseline system is significantly enhanced as SNR increases and the crossover of the two curves appears at around 20 dB, which is caused by the loss in the spectral efficiency of the half-duplex relay system. Same trend is observed for average PSNR. In Figs. 5 and 6, average code rates of UEP policies for the two systems are presented. Fig. 5 shows the effective code rate, $r_c$, where the effective code rate is defined as the ratio of the number of information bits per packet to the packet length. In the half-duplex relay system, the packet consists of two time slots, where the second time slot is the repetition of the first slot if relay decodes the first time slot successfully. Therefore, even though the two systems have the same channel code rates, the number of information bits per packet in the half-duplex relay system is half of that in the baseline system, and the effective code rate of the half duplex relay system cannot be greater than $1/2 \max\{R\}$, as shown in Fig. 5. Interestingly, the average code rates for the half-duplex relay system is fixed at $4/9$ after 12 dB of channel SNR. This is mainly caused by the high diversity gain of the relay system, which allows for a higher code rate. Fig. 6 presents the average RCPC and RCPT code rates, where the code rate of the half-duplex relay system represents the code rate per time slot, not per packet. As shown in this figure, the average code

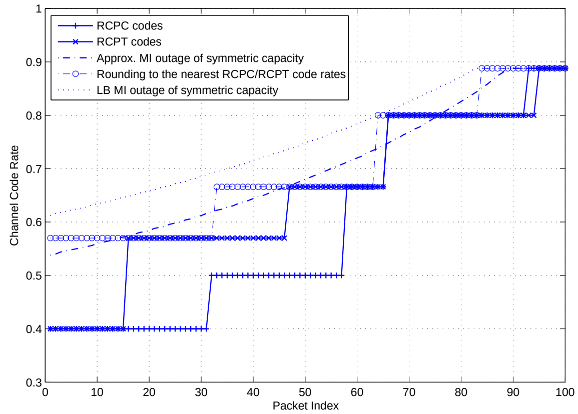


Fig. 7. Channel code rate profile of the baseline system at 18 dB

rates corresponding to the half duplex relay system converge to $\max\{R\}$ at an SNR around 14 dB. That is, when channel SNR is greater than 14 dB, half duplex relay system can choose the highest code rate for all packets and then rate-optimal UEP becomes EEP. Finally, the rate-optimal UEP profiles of the baseline system at an SNR of 18 dB are presented in Fig. 7. The UEP profiles of the RCPC and RCPT codes are found using the corresponding PER and the algorithm of [8], while two above curves represent UEP profiles computed by using approximated MI outage probability and its lower bound. The stepwise UEP profiles are also presented by rounding the UEP profiles of the MI outage probability to the nearest RCPC and RCPT code rates. As can be seen, in most packets, the analytically-derived UEP channel code rates are higher than the UEP channel code rates of the RCPC and RCPT codes, since the MI outage probability lower bounds the PER. This can help in finding the rate-optimal UEP for a specific channel coding scheme by reducing the possible code-rate set.

## VI. Conclusion

We studied source channel rate allocation for a general progressive image transmission system over a block fading relay channel. In particular, we derived the outage probability for the baseline and the half-duplex relay systems with Gaussian as well as BPSK inputs. We then derived the average throughput expressions using the channel MI outage probability, which is a lower bound on the PER. To solve the optimization problem, we proved the concavity of the objective functions at high SNR and then found the solution using the recursive algorithm from [8]. We compared the average throughput and the rate-optimal UEP code rates found from the analysis with those obtained from simulations of a BPSK-based system. Numerical results indicated that our information-theoretic system model yielded an approximate upper bound at high SNR on the system performance for the actual BPSK-based system with capacity achieving channel coding.

TABLE I
MI Outage Probabilities of Baseline System

| | |
|---|---|
| General Form | $\pi_i = \gamma_{\text{inc}}(\frac{1}{\Gamma}f(r_i), N)$ |
| Gaussian Inputs | $f(r_i) = 2^{r_i} - 1$ |
| BPSK Approx. | $f(r_i) = -\frac{\ln(1-r_i)}{2b}$ |
| BPSK Upper Bound (UB) | $f(r_i) = -\frac{\ln(2^{1-r_i}-1)}{2}$ |

## Appendix A
## Concavity of Average Throughput Expression for the Baseline System

In this appendix, we demonstrate the concavity of average throughput for the baseline system when the SNR $>> 1$. Ignoring constants, the average throughput expression of the baseline system is

$$\overline{\mathcal{T}}(\underline{r}_c) = \sum_{l=1}^{M}(\sum_{k=1}^{l}\tilde{r}_c(k))\mathsf{S}_l = \sum_{l=1}^{M}\tilde{r}_c(l)\prod_{k=1}^{l}(1-\pi_k). \quad (36)$$

As can be seen in [8], for the rate-optimal UEP solution of a progressive image, the following nondecreasing condition holds : $\tilde{r}_c(1) \le \tilde{r}_c(2) \le \cdots \le \tilde{r}_c(M)$. Therefore, we can focus on the region of $\underline{r}_c$ which satisfies the nondecreasing condition for the concavity proof. To generate comparable results, we also restrict the range of $\underline{r}_c$ to be between $\min\{R\}$ and $\max\{R\}$, where $\{R\}$ is the set of possible rates for the RCPC and RCPT codes. For simplicity of notation, we replace $\overline{\mathcal{T}}(\underline{r}_c)$, $\tilde{r}_c(i)$, $\frac{d\pi_i}{dr_i}$, and $\frac{d^2\pi_i}{dr_i^2}$ by $\overline{\mathcal{T}}$, $r_i$, $\pi_i'$ and $\pi_i''$. From (36), we can derive the second and cross partial derivatives of $\overline{\mathcal{T}}(\underline{r}_c)$. Since $\pi_i$ is a function of $r_i$, the second order and cross partial derivatives of $\overline{\mathcal{T}}$ are

$$\frac{\partial^2\overline{\mathcal{T}}}{\partial {r_i}^2} = -2\pi_i'\prod_{k=1}^{i-1}(1-\pi_k) - \sum_{l=i}^{M} r_l\pi_i''\prod_{\substack{k=1\\k\neq i}}^{l}(1-\pi_k) \quad (37)$$

and

$$\frac{\partial^2\overline{\mathcal{T}}}{\partial r_i\partial r_j} = -\pi_i'\prod_{\substack{k=1\\k\neq i}}^{j}(1-\pi_k) + \sum_{l=j}^{M} r_l\pi_i'\pi_j'\prod_{\substack{k=1\\k\neq i,j}}^{l}(1-\pi_k) \quad (38)$$

respectively, where $i < j$. The MI outage probability, $\pi_i$, for both Gaussian and BPSK inputs, are presented in Table I. Note that MI outage probability for BPSK inputs are approximated to get closed form expression. The first and second order derivatives of $\pi_i$, where

$$\pi_i = \frac{1}{(N-1)!}\int_0^{\frac{1}{\Gamma}f(r_i)} e^{-u}u^{N-1}du, \quad (39)$$

are given by

$$\pi_i' = \left(\frac{1}{\Gamma}\right)^N e^{-\frac{1}{\Gamma}f(r_i)}\left(f(r_i)\right)^{N-1}f'(r_i)\frac{1}{(N-1)!} \quad (40)$$

and (41), respectively.

$$\pi_i'' = \left(\frac{1}{\overline{\Gamma}}\right)^N e^{-\frac{1}{\Gamma}f(r_i)} \left(f(r_i)\right)^{N-2} \frac{1}{(N-1)!}\left( -\frac{1}{\overline{\Gamma}} f(r_i)\left(f'(r_i)\right)^2 + (N-1)\left(f'(r_i)\right)^2 + f(r_i)f''(r_i)\right) \tag{41}$$

Since $f(r_i)$ and $f'(r_i)$ are positive for all inputs, $\pi_i$ and $\pi_i'$ are positive, too. Moreover, $\pi_i''$ is also positive if

$$\overline{\Gamma} > \begin{cases} \dfrac{2^{r_i}(2^{r_i}-1)}{N2^{r_i}-1}, & \text{for Gaussian inputs} \\ \dfrac{\ln(1-r_i)}{2b\left(\ln(1-r_i)-N+1\right)}, & \text{for BPSK approx.} \\ \dfrac{\ln(2^{1-r_i}-1)}{2\left(2^{r_i-1}\ln(2^{1-r_i}-1)-N+1\right)}, & \text{for BPSK UB.} \end{cases}$$

From (38), the cross derivative of average throughput is given by (42). Note that $f(r_i)$ and $f'(r_i)$ are limited by $f(\max\{R\})$ and $f'(\max\{R\})$. Therefore, if $\overline{\Gamma}$ is large enough, (38) will be negative since its first term will become dominant. Next, we prove the following inequalities :

$$-\frac{1}{2}\frac{\partial^2\overline{\mathcal{T}}}{\partial {r_i}^2} + \frac{\partial^2\overline{\mathcal{T}}}{\partial r_i \partial r_j} > 0 \tag{43}$$

$$-\frac{1}{2}\frac{\partial^2\overline{\mathcal{T}}}{\partial {r_j}^2} + \frac{\partial^2\overline{\mathcal{T}}}{\partial r_i \partial r_j} > 0. \tag{44}$$

Eqs. (43) and (44) can be shown by (45) and (46), where $\pi_j$ and $\pi_j'$ are monotonic increasing functions of $r_i$ and

$$\begin{aligned} \pi_j'(1-\pi_i) - \pi_i'(1-\pi_j) &> (\pi_j' - \pi_i') + \pi_j(\pi_i' - \pi_j') \\ &= (1-\pi_j)(\pi_j' - \pi_i') > 0. \end{aligned}$$

Now consider the Hessian matrix of the objective function. Note that the negative definiteness of the Hessian matrix implies the strict concavity of the objective function. Let $H_m$ denote the $m$th principal submatrix of the Hessian matix, which is computed from the objective function, $\overline{\mathcal{T}}$. We have shown that all elements of the Hessian matrix, (37) and (38), are negative, and (43) and (44) are hold. Therefore, for any positive integer $i$ and $j$ less than or equal to $M$, we can represent $-\frac{\partial^2\overline{\mathcal{T}}}{\partial {r_i}^2} = 2a_i > 0$ and $-\frac{\partial^2\overline{\mathcal{T}}}{\partial r_i \partial r_j} = a_i - \delta_{ij} > 0$, where $a_i > 0$ and $0 < \delta_{ij} < a_i$. Then, $H_m$ can be simplified as in (47). If we define $-G_m$ as in (48), where $\eta_{ij} = a_m - \frac{a_m}{a_j}(a_i - \delta_{ij}) > 0$, then

$$\det(-H_m) = \frac{a_1}{a_m}\frac{a_2}{a_m}\dots\frac{a_{m-1}}{a_m}\det(-G_m) \tag{49}$$

and the sign of $\det(-H_m)$ is equivalent to that of $\det(-G_m)$. To evaluate the sign of $\det(-G_m)$, we use the following theorem introduced in [24] and lemmas.

*Theorem 1:* If all the principal minors of a matrix are positive and all the elements off its main diagonal are negative, then all the elements of its inverse are positive [24].

*Lemma 1:* If $A$ is a positive definite matrix whose elements are all positive, and $Z$ is an all-zero matrix except it has an off diagonal positive element in the $i$th row and $j$th column which is less than the $(i, j)$th element of $A$, then $A - Z$ is also a positive definite matrix.

*Proof:* Let's consider all principal submatrices of $A - Z$. Since only the $(i, j)$th element differs between $(A - Z)$ and $A$, up to the $\max(i-1, j-1)$th principal submatrix, their determinants are unchanged. If we assume $i < j$, then the $j$th and subsequent principal submatrices will have different determinants. Denote the $j$th principal submatrix of $A$ as $A_j$ and that of $Z$ as $Z_j$. Then, we can define a matrix, $X_j$, where $X_j$ is an all zero matrix except for its $i$th row, and satisfies

$$X_j = Z_j A_j^{-1}. \tag{50}$$

From Theorem 1, all elements of $A_j^{-1}$ are negative except its diagonal element. Since there is only one positive element in $Z_j$, the nonzero diagonal element of $X_j$, $x_{ii}$, is negative. Then,

$$A_j - Z_j = (I - X_j)A_j \tag{51}$$

and

$$\det(A_j - Z_j) = \det(I - X_j)\det(A_j) = (1 - x_{ii})\det(A_j) > 0. \tag{52}$$

Therefore, the determinant of the $j$th principal submatrix is positive. In the same way, we can show the positiveness of all submatrices' determinants. Since all submatrices have positive determinant, the new matrix, $A - Z$, is positive definite. ∎

*Lemma 2:* The following matrix is positive definite for any positive number $a_m$ :

$$F = \begin{pmatrix} 2a_m & a_m & a_m & \dots & a_m \\ a_m & 2a_m & a_m & \dots & a_m \\ a_m & a_m & 2a_m & \dots & a_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_m & a_m & a_m & \dots & 2a_m \end{pmatrix}.$$

*Proof:* For the $i$th principal submatrix of $F$, $F_i$,

$$\det(F_i) = (a_m)^m 2\frac{3}{2}\frac{4}{3}\dots\frac{i+1}{i} > 0 \tag{53}$$

∎

With Lemma 1 and 2, $-G_m$ is positive definite, which implies that both $\det(-G_m)$ and $\det(-H_m)$ are positive for all $m$. Therefore, the Hessian matrix is negative definite and the average throughput is a concave function over $\underline{r}$.

## APPENDIX B
## CONCAVITY OF THE AVERAGE THROUGHPUT EXPRESSION FOR THE HALF-DUPLEX RELAY SYSTEM

The average throughput expression for the half-duplex relay system is equivalent to that for the baseline system except for its packet error probability. Instead of following all steps in the previous appendix, we show the following inequalities for

$$\frac{\partial^2 \overline{\mathcal{T}}}{\partial r_i \partial r_j} = -\left(\frac{1}{\overline{\overline{\Gamma}}}\right)^N \frac{1}{(N-1)!} e^{-\frac{1}{\overline{\Gamma}} f(r_i)} \left(f(r_i)\right)^{N-1} f'(r_i) \prod_{\substack{k=1 \\ k\neq i}}^{j} (1-\pi_k) + \left(\frac{1}{\overline{\overline{\Gamma}}}\right)^{2N} \left(\frac{1}{(N-1)!}\right)^2 \cdot$$
$$e^{-\frac{1}{\Gamma}(f(r_i)+f(r_j))} \left(f(r_i) f(r_j)\right)^{N-1} f'(r_i) f'(r_j) \sum_{l=j}^{M} r_l \prod_{\substack{k=1 \\ k\neq i,j}}^{l} (1-\pi_k) \tag{42}$$

---

$$-\frac{1}{2}\frac{\partial^2 \overline{\mathcal{T}}}{\partial {r_i}^2} + \frac{\partial^2 \overline{\mathcal{T}}}{\partial r_i \partial r_j} = \pi_i' \prod_{k=1}^{i-1}(1-\pi_k)\left(1 - \prod_{k=i+1}^{j}(1-\pi_k)\right) + \frac{1}{2}\sum_{l=i}^{M} r_l \pi_i'' \prod_{\substack{k=1 \\ k\neq i}}^{l}(1-\pi_k) + \sum_{l=j}^{M} r_l \pi_i' \pi_j' \prod_{\substack{k=1 \\ k\neq i,j}}^{l}(1-\pi_k) > 0 \tag{45}$$

---

$$-\frac{1}{2}\frac{\partial^2 \overline{\mathcal{T}}}{\partial {r_j}^2} + \frac{\partial^2 \overline{\mathcal{T}}}{\partial r_i \partial r_j} = \left(\pi_j'(1-\pi_i) - \pi_i'(1-\pi_j)\right) \prod_{\substack{k=1 \\ k\neq i}}^{j-1}(1-\pi_k) + \frac{1}{2}\sum_{l=j}^{M} r_l \pi_j'' \prod_{\substack{k=1 \\ k\neq j}}^{l}(1-\pi_k) + \sum_{l=j}^{M} r_l \pi_i' \pi_j' \prod_{\substack{k=1 \\ k\neq i,j}}^{l}(1-\pi_k) > 0 \tag{46}$$

---

$$-H_m = \begin{pmatrix} 2a_1 & a_1-\delta_{12} & a_1-\delta_{13} & \dots & a_1-\delta_{1m} \\ a_1-\delta_{21} & 2a_2 & a_2-\delta_{23} & \dots & a_2-\delta_{2m} \\ a_1-\delta_{31} & a_2-\delta_{32} & 2a_3 & \dots & a_3-\delta_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1-\delta_{m1} & a_2-\delta_{m2} & a_3-\delta_{m3} & \dots & 2a_m \end{pmatrix} \tag{47}$$

$$-G_m = \begin{pmatrix} 2a_m & a_m-\eta_{12} & a_m-\eta_{13} & \dots & a_m-\eta_{1m} \\ a_m-\eta_{21} & 2a_m & a_m-\eta_{23} & \dots & a_m-\eta_{2m} \\ a_m-\eta_{31} & a_m-\eta_{32} & 2a_m & \dots & a_m-\eta_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_m-\eta_{m1} & a_m-\eta_{m2} & a_m-\eta_{m3} & \dots & 2a_m \end{pmatrix} \tag{48}$$

---

the half duplex relay system :

$$\frac{\partial^2 \overline{\mathcal{T}}}{\partial {r_i}^2} < 0 \tag{54}$$
$$\frac{\partial^2 \overline{\mathcal{T}}}{\partial r_i \partial r_j} < 0 \tag{55}$$
$$-\frac{1}{2}\frac{\partial^2 \overline{\mathcal{T}}}{\partial r_i \partial r_j} + \frac{\partial^2 \overline{\mathcal{T}}}{\partial {r_i}^2} > 0 \tag{56}$$
$$-\frac{1}{2}\frac{\partial^2 \overline{\mathcal{T}}}{\partial r_i \partial r_j} + \frac{\partial^2 \overline{\mathcal{T}}}{\partial {r_j}^2} > 0, \tag{57}$$

where the superscript $'\mathrm{CoOp}'$ was dropped throughout this appendix and $\overline{\mathcal{T}}_{\mathrm{CoOp}}(\underline{r}_c^{\mathrm{CoOp}})$, $r_c^{\mathrm{CoOp}}(i)$, and the first and second partial derivatives of $\pi_i^{CoOp}$ are denoted by $\overline{\mathcal{T}}$, $r_i$, $\pi_i'$, and $\pi_i''$, respectively. Then, the remainder of the proof is equivalent to that of the baseline system. As shown in (24), the average error probability for the $j$th packet in the relay system is given by

$$\pi_j = \sum_{\mathcal{D}_j} \mathrm{Prob}(\mathcal{D}_j)\pi_j(\mathcal{D}_j), \tag{58}$$

where $\mathrm{Prob}(\mathcal{D}_j)$ is the probability that the relay nodes in a set, $\mathcal{D}_j$, decode the $j$th source packet successfully, and $\pi_j(\mathcal{D}_j)$ is the probability that the $j$th packet is in error at the destination, conditioned on $\mathcal{D}_j$. With the high SNR assumption and the fact that, for small $x$, $\exp(-x) \approx 1-x$, $\mathrm{Prob}(\mathcal{D}_j)$ and $\pi_j(\mathcal{D}_j)$ can be approximated as [25]

$$\begin{aligned} \mathrm{Prob}(\mathcal{D}_j) &= \left\{\prod_{i\in\mathcal{D}_j}\left(1-\gamma_{\mathrm{inc}}\left(\frac{f(r_j)}{\overline{\gamma}_{sr}^{i} N}, 1\right)\right)\right\} \\ &\quad \times \left\{\prod_{k\notin\mathcal{D}_j}\gamma_{\mathrm{inc}}\left(\frac{f(r_j)}{\overline{\gamma}_{sr}^{k} N}, 1\right)\right\} \\ &\approx \prod_{k\notin\mathcal{D}_j}\frac{f(r_j)}{\overline{\gamma}_{sr}^{k} N} \end{aligned} \tag{59}$$

and

$$\begin{aligned} &\pi_j(\mathcal{D}_j) \\ &= \mathrm{Prob}\left(\frac{\overline{\gamma}_{sd}(\alpha_{sd,j})^2}{\Omega_{sd}} + \sum_{l\in\mathcal{D}_j}\frac{\overline{\gamma}_{rd}^{l}(\alpha_{rd,j}^{l})^2}{\Omega_{rd}^{l}} < \frac{f(r_j)}{N}\right) \\ &\approx \frac{1}{(|\mathcal{D}_j|+1)!}\left(\frac{f(r_j)}{N}\right)^{|\mathcal{D}_j|+1}\frac{1}{\overline{\gamma}_{sd}}\prod_{l\in\mathcal{D}_j}\frac{1}{\overline{\gamma}_{rd}^{l}}. \end{aligned} \tag{60}$$

$$\pi_j \approx \sum_{\mathcal{D}_j} \frac{1}{(|\mathcal{D}_j|+1)!} \left(\frac{f(r_j)}{N}\right)^N \frac{1}{\overline{\gamma}_{sd}} \prod_{k \notin \mathcal{D}_j} \frac{1}{\overline{\gamma}_{sr}^k} \prod_{l \in \mathcal{D}_j} \frac{1}{\overline{\gamma}_{rd}^l} \tag{61}$$

$$\pi_j' \approx \left(\frac{f(r_j)}{N}\right)^{N-1} f'(r_j) \sum_{\mathcal{D}_j} \frac{1}{(|\mathcal{D}_j|+1)!} \frac{1}{\overline{\gamma}_{sd}} \prod_{k \notin \mathcal{D}_j} \frac{1}{\overline{\gamma}_{sr}^k} \prod_{l \in \mathcal{D}_j} \frac{1}{\overline{\gamma}_{rd}^l} \tag{62}$$

$$\pi_j'' \approx \frac{(f(r_j))^{N-2}}{N^{N-1}} \left((N-1)\left(f'(r_j)\right)^2 + f(r_j) f''(r_j)\right) \sum_{\mathcal{D}_j} \frac{1}{(|\mathcal{D}_j|+1)!} \frac{1}{\overline{\gamma}_{sd}} \prod_{k \notin \mathcal{D}_j} \frac{1}{\overline{\gamma}_{sr}^k} \prod_{l \in \mathcal{D}_j} \frac{1}{\overline{\gamma}_{rd}^l} \tag{63}$$

TABLE II
DECODING ERROR PROBABILITY OF HALF-DUPLEX RELAY SYSTEM

| | |
|---|---|
| General Form | $q(i,j) = \gamma_{\text{inc}}(\frac{1}{\overline{\gamma}_{sr}^i(N)} f(r_i), 1)$ |
| Gaussian Inputs | $f(r_i) = 2^{2r_i} - 1$ |
| BPSK Approx. | $f(r_i) = -\frac{\ln(1-2r_i)}{2b}$ |
| BPSK Upper Bound (UB) | $f(r_i) = -\frac{\ln(2^{1-2r_i}-1)}{2}$ |

With (59) and (60), $\pi_j$ and its derivatives are approximated as in (61), (62), and (63). and Note that $f'(r_j)$ and $f''(r_j)$ are positive for any inputs. Therefore, the approximations of both $\pi_j'$ and $\pi_j''$ are positive, and (54), (56), and (57) can be proved as shown in the previous appendix. To prove the remaining inequality, we define

$$L_j = \sum_{\mathcal{D}_j} \frac{1}{(|\mathcal{D}_j|+1)!} \frac{1}{\overline{\gamma}_{sd}} \prod_{k \notin \mathcal{D}_j} \frac{1}{\overline{\gamma}_{sr}^k} \prod_{l \in \mathcal{D}_j} \frac{1}{\overline{\gamma}_{rd}^l}.$$

Then, the approximation of $\frac{\partial^2 \overline{\mathcal{T}}}{\partial r_i \partial r_j}$ is shown in (64), where $L_i$ is equivalent to $L_j$, since the possible sets of $\mathcal{D}_j$ are the same as the possible sets of $\mathcal{D}_i$. Since $f(r_i)$ and $f'(r_i)$ are limited by $f(\max\{R\})$ and $f'(\max\{R\})$, the second term will be relatively small if the SNR is high enough, and then

$$\frac{\partial^2 \overline{\mathcal{T}}}{\partial r_i \partial r_j} \approx -L_i \left(\frac{f(r_i)}{N}\right)^{N-1} f'(r_i) \prod_{k=1, k\neq i}^{j} (1-\pi_k) < 0.$$

## APPENDIX C
## OPTIMIZATION OF SOURCE-CHANNEL RATE ALLOCATION

Due to the concavity of the average throughput expression, the optimal UEP is the peak of the expression, if it exists. The peak can be found by $M$ partial differentiations as follows :

$$\frac{\partial \overline{\mathcal{T}}}{\partial r_i} = 0 \quad i = 1, \ldots, M. \tag{65}$$

If there is no solution for the $i$th equation, then the $i$th element of the optimal UEP is $\min\{R\}$ or $\max\{R\}$. In particular, $\frac{\partial \overline{\mathcal{T}}}{\partial r_M}$ will be

$$\frac{\partial \overline{\mathcal{T}}}{\partial r_M} = (1-\pi_M) - r_M \pi_M' = 0. \tag{66}$$

That is, we can find the optimum rate of the last packet first and then find the rate of the $(M-1)$th packet recursively. This recursive algorithm is equivalent to that introduced in [8].

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$$\frac{\partial^2 \overline{\mathcal{T}}}{\partial r_i \partial r_j} \approx -L_i \left(\frac{f(r_i)}{N}\right)^{N-1} f'(r_i) \prod_{k=1,k\neq i}^{j} (1-\pi_k) + L_i^2 \left(\frac{f(r_i)}{N}\right)^{N-1} f'(r_i) \left(\frac{f(r_j)}{N}\right)^{N-1} f'(r_j) \sum_{l=j}^{M} r_l \prod_{k=1,k\neq i,j}^{l} (1-\pi_k) \tag{64}$$


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