On the Output Transformation To An Observer Form

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Abstract

This paper investigates the transformability of an unforced multi-output nonlinear system to a multi-output observer form. The existence conditions of an output transformation and a change of state coordinates are presented in a more concise form than those given in literatures. Given an output transformation, verifying these conditions can reveal if the unforced system is transformable to the observer form. Necessary conditions on the output transformation are given for the single output and multi-output nonlinear systems. These necessary conditions are stated as a set of first order partial differential equations, which are relatively easy to solve and potentially useful to obtain the output transformation candidates.

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***Abstract*— This paper investigates the transformability of an unforced multi-output nonlinear system to a multi-output observer form. The existence conditions of an output transformation and a change of state coordinates are presented in a more concise form than those given in literatures. Given an output transformation, verifying these conditions can reveal if the unforced system is transformable to the observer form. Necessary conditions on the output transformation are given for the single output and multi-output nonlinear systems. These necessary conditions are stated as a set of first order partial differential equations, which are relatively easy to solve and potentially useful to otain the output transformation candidates.**

I. Introduction

We consider observer design for uncontrolled multi-output systems in state space form

$$\begin{aligned}\dot{\zeta} &= f(\zeta),\\ y &= h(\zeta)\end{aligned} \tag{1}$$

where $\dot{\zeta}$ denotes $\mathrm{d}\zeta/\mathrm{d}t$, $\zeta = (\zeta_1, \cdots, \zeta_n)^T \in \mathbb{R}^n$ is the state, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^∞ vector field, and $h : \mathbb{R}^n \to \mathbb{R}^p$ is a C^∞ output function. The well-established exact error linearization nonlinear observer design method uses an Observer Form (OF) to obtain stable Linear Time-Invariant (LTI) state estimate error dynamics in OF coordinates [14], [2]. Significant effort has been placed on extending this original work for single-output continuous-time systems [15], [25], [7], [23], [18], [11], [13], [19], [16], [3]. Some of the extensions are achieved by eliminating constraints in the target normal forms. For instance, the block triangular observer form in [23] allows a more general dependence in the system's output injection vector. Other approaches apply immersion techniques or dynamic error linearization [17], [22], [1].

Given the wide array of nonlinear observer design methods that have been developed, OF-based methods benefit from

- a relatively straightforward design procedure based on normal form coordinates
- potentially larger regions of attraction.

Albeit for a different class of systems, these attributes should be compared favorably with those of alternatives [11], [13], [5].

In Section II we recall some fundamental concepts and state the problem to be discussed. The existence conditions of an output transformation and a change of state coordinates which transform a multi-output system into the observer form are given in Section III. Necessary conditions for the output transformation is presented to address the non-constructive nature of the existence conditions in Section IV. The application of the proposed result to a perspective system is illustrated in Section V. Conclusion is made in VI.

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II. Problem Statement

A. Background and Notation

Given a C^∞ vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, and a C^∞ function $\alpha : \mathbb{R}^n \to \mathbb{R}$, the function $L_f\alpha = \frac{\partial \alpha}{\partial \zeta} f$ is the *Lie derivative* of α along f. The *differential* or *gradient* of a C^∞ function $\alpha : \mathbb{R}^n \to \mathbb{R}$ is denoted $\mathrm{d}\alpha$ and has local coordinate description $\mathrm{d}\alpha = \frac{\partial \alpha}{\partial \zeta} = (\frac{\partial \alpha}{\partial \zeta_1}, \ldots, \frac{\partial \alpha}{\partial \zeta_n})$. Given a C^∞ one-form $\omega : \mathbb{R}^n \to \mathbb{R}^n$ and a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, the *inner product* of ω and f is the function $\langle \omega(\zeta), f(\zeta) \rangle = \sum_{i=1}^n \omega_i(\zeta) f_i(\zeta)$, where ω_i, f_i are the components of ω, f in local coordinates, respectively. The *Lie bracket* of two C^∞ vector fields $f, g : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$[f, g] = \frac{\partial g}{\partial \zeta} f - \frac{\partial f}{\partial \zeta} g.$$

Given two smooth functions α, β and two smooth vector fields f, g, the following formula holds:

$$[\alpha f, \beta g] = \alpha\beta[f, g] + \alpha L_f(\beta) g - \beta L_g(\alpha) f.$$

Repeated Lie brackets are defined as $\mathrm{ad}_f^k g = [f, \mathrm{ad}_f^{k-1} g]$, $k \geq 1$ with $\mathrm{ad}_f^0 g = 0$. See [8], [21] for further details.

I_k denotes a $k \times k$ identity matrix. $\delta_{j,k}$ is the Kronecker delta. Other notation include $h(\cdot) = (h_1(\cdot), \ldots, h_p(\cdot))^T$, $y = (y_1, \ldots, y_p)^T$, and $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_p)^T$. For simplicity, we abbreviate $h(\cdot)$ to h.

B. Problem Statement

We first introduce the definition of observability. Different notions of observability exist [6], [20], [23]. We take the definition which is based on uniquely defined observability indices and ensures a single normal form. We recite the definition in [20].

Definition 2.1: System (1) is locally observable in U_0 with observability indices $\lambda_i, 1 \leq i \leq p$, if

$$\dim(\mathrm{span}\{L_f^{j-1}\mathrm{d}h_i(\zeta),\ 1 \leq j \leq \lambda_i;\ 1 \leq i \leq p\}) = n, \tag{2}$$

for every $\zeta \in U_0$. □

Remark 2.2: System (1) is globally observable if $L_f^{j-1} h_i, 1 \leq j \leq \lambda_i, 1 \leq i \leq p$ are globally defined coordinates on $\mathbb{R}^n$. For the mapping

$$x = T(\zeta) = (h_1, \cdots, L_f^{\lambda_1 - 1} h_1, \cdots, L_f^{\lambda_p - 1} h_p)^T$$

to be a global diffeomorphism, in addition to satisfying (2) for all $\zeta \in \mathbb{R}^n$, it also requires condition [24], [12]

$$\lim_{\|\zeta\| \to \infty} \|T(\zeta)\| = \infty$$

holds. □

Problem 2.3: Given the locally (globally) observable unforced nonlinear system (1), find the existence conditions of an output transformation $\bar{y} = \Psi(y)$ and a local (global) diffeomorphism $z = \Phi(\zeta)$ s.t. system is locally (globally) transformable to Observer Form (OF)

$$\begin{aligned} \dot{z} &= Az + \gamma(\bar{y}), \\ \bar{y} &= Cz, \end{aligned} \tag{3}$$

where matrices $A \in \mathbb{R}^{n\times n}, C \in \mathbb{R}^{p\times n}$ are block diagonal

$$\begin{aligned} A &= \text{blockdiag}(A^1, \ldots, A^p), \\ C &= \text{blockdiag}(C^1, \ldots, C^p), \end{aligned} \tag{4}$$

and each pair

$$A^i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{\lambda_i \times \lambda_i},$$
$$C^i = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{1\times\lambda_i}$$

is in dual Brunovsky Form, and $\lambda_i, 1 \le i \le p$ are observability indices in [20].

III. Existence Conditions

We introduce two co-distributions Q_i, Q [25]

$$\begin{aligned} Q_i &= \text{span}\{\mathrm{d}L_f^k h_r(\zeta), 0 \le k \le \lambda_i - 1, 1 \le r \le p\} \\ &\quad \backslash \{\mathrm{d}L_f^{\lambda_i - 1} h_i(\zeta)\}, \quad 1 \le i \le p, \\ Q &= \text{span}\{\mathrm{d}L_f^k h_r(\zeta), 0 \le k \le \lambda_r - 1, 1 \le r \le p\}. \end{aligned}$$

The existence conditions of (3) are given by the following theorem.

Theorem 3.1: There exists an output transformation $\bar{y} = \Psi(y)$ and a local diffeomorphism $z = \Phi(\zeta)$, transforming system (1) into an OF (3) if, and only if, in U_0

1) the system is locally observable and $\lambda_1, \cdots, \lambda_p$ are constant;
2) $Q_i = Q_i \bigcap Q$.
3) the vector fields $g^1, \cdots, g^p$ satisfying

$$L_{\bar{g}^i} L_f^k h_j = \alpha_j^i(y) \delta_{i,j} \delta_{k,\lambda_i - 1}, \quad \begin{cases} 1 \le i, j \le p; \\ 0 \le k \le \lambda_i - 1, \end{cases} \tag{5}$$

4) The matrix

$$\alpha(y) = \begin{bmatrix} \alpha_1^1 & \cdots & \alpha_1^p \\ \vdots & \ddots & \vdots \\ \alpha_p^1 & \cdots & \alpha_p^p \end{bmatrix} = (\alpha^1, \ldots, \alpha^p) \tag{6}$$

is nonsingular, and the Lie bracket condition holds

$$[\alpha^i, \alpha^j] = 0, \quad 1 \le i, j \le p. \tag{7}$$

5) Lie bracket condition holds, i.e.,

$$\left[\mathrm{ad}_{-f}^k \bar{g}^i, \mathrm{ad}_{-f}^l \bar{g}^j\right] = 0, \quad \begin{cases} 1 \le i, j \le p; \\ 0 \le k \le \lambda_i - 1; \\ 0 \le l \le \lambda_j - 1. \end{cases} \tag{8}$$

The output transformation $\Psi(y)$ is obtained by solving PDEs

$$\frac{\partial \Psi(y)}{\partial y} \alpha(y) = I_p. \tag{9}$$

The state transformation $\Phi(\zeta)$ is obtained by solving PDEs

$$\frac{\partial \Phi(\zeta)}{\partial \zeta} \left(\mathrm{ad}_{-f}^{\lambda_1 - 1} \bar{g}^1, \cdots, \bar{g}^1, \cdots, \mathrm{ad}_{-f}^{\lambda_p - 1} \bar{g}^p, \ldots, \bar{g}^p\right) = I_n. \tag{10}$$

Remark 3.2: There exists a global diffeomorphism if, and only if, Conditions (1)–(5) holds in $\mathbb{R}^n$ and, in addition, the vector fields

$$\begin{aligned} &\alpha^k, \quad 1 \le k \le p, \\ &\mathrm{ad}_{-f}^k \bar{g}^i, \quad 0 \le k \le \lambda_i - 1; 1 \le i \le p \end{aligned}$$

are complete. □

Remark 3.3: Condition 4) in Theorem 3.1 ensures that an output transformation $\Psi(y)$ can be solved. Particularly, (7) guarantees the solvability of the PDE system (10), and (6) is necessary for $z = \Phi(\zeta)$ to be a local diffeomorphism. □

The introduction of an output transformation changes the definition of the starting vector g^i, and results in extra conditions (6) and (7). The following proof therefore only shows the necessity of these additional conditions. For the proof of the rest conditions, readers are referred to [25], [20].

Proof: Assume system (1) is transformed into (3) by a change of state coordinates $z = \Phi(\zeta)$ and an output transformation $\bar{y} = \Psi(y)$. The original output, in z-coordinates, is expressed as

$$y = \beta(\bar{y}_1, \ldots, \bar{y}_p) = \beta(z_1^1, \ldots, z_1^p).$$

Evidently, $\beta(\bar{y}) = \Psi^{-1}(\bar{y})$. This implies that in the z-coordinates

$$\frac{\partial y}{\partial z} = \left[\frac{\partial y}{\partial z^1}, \ldots, \frac{\partial y}{\partial z^p}\right],$$

where $z^i = (z_1^i, \ldots, z_{\lambda_i}^i)^T$, and

$$\frac{\partial y}{\partial z^i} = \begin{bmatrix} \alpha_1^i & 0 \\ \vdots & \vdots \\ \alpha_p^i & 0 \end{bmatrix} \in \mathbb{R}^{p\times\lambda_i}. \tag{11}$$

It is clear that (11) implies conditions about the starting vector $\bar{g}^i$. We rewrite (11) as

$$\left\langle \mathrm{d}y_1, \frac{\partial}{\partial z_1^i} \right\rangle = \alpha_1^i, \quad \left\langle \mathrm{d}y_1, \frac{\partial}{\partial z_k^i} \right\rangle = 0, \quad 2 \le k \le \lambda_i,$$
$$\vdots$$
$$\left\langle \mathrm{d}y_p, \frac{\partial}{\partial z_1^i} \right\rangle = \alpha_p^i, \quad \left\langle \mathrm{d}y_p, \frac{\partial}{\partial z_k^i} \right\rangle = 0, \quad 2 \le k \le \lambda_i.$$

According to the expression of (3), it is easy to verify that

$$\frac{\partial}{\partial z^i_{\lambda_i}} = \bar{g}^i, \cdots, \frac{\partial}{\partial z^i_1} = \mathrm{ad}^{\lambda_i - 1}_{-f}\bar{g}^i.$$

Applying [20, Thm. A.3.1], we conclude

$$\begin{aligned} &L_{\bar{g}^i} L_f^{\lambda_i-1} y_1 = \alpha^i_1,\ L_{\bar{g}^i} L_f^k y_1 = 0,\ 0 \le k \le \lambda_i - 2, \\ &\qquad\vdots \\ &L_{\bar{g}^i} L_f^{\lambda_i-1} y_p = \alpha^i_p,\ L_{\bar{g}^i} L_f^k y_p = 0,\ 0 \le k \le \lambda_i - 2, \end{aligned}$$

which implies that (5) holds in z-coordinates. Since (5) is independent of coordinates, we therefore prove the necessity of (5).

As for the necessity of (6), since

$$(z^1_1, \ldots, z^p_1)^T = \Psi(y) = \bar{y},$$

we have

$$\frac{\partial \Psi(y)}{\partial y}\frac{\partial y}{\partial \bar{y}} = I_p.$$

Evidently, $\alpha(y) = \frac{\partial y}{\partial \bar{y}}$, we have

$$\frac{\partial \Psi(y)}{\partial y}\alpha(y) = I_p.$$

Hence, it is necessary for $\alpha(y)$ to be nonsingular, and the output transformation is solved from (9). To solve the output transformation from (9), it is equivalent to rectify the vector fields α^i into unit vectors in the p-dimensional output space. According to the Simultaneous Rectification Theorem [21], it requires that the vector fields α^i commute, i.e., (7) should hold. ■

Remark 3.4: When applying Theorem 3.1 to transform system (1) into (3), the key is to construct the starting vector $\bar{g}^k, 1 \le k \le p$. Due to Condition 2) in Theorem 3.1, the local observability of system (1) does not guarantee the solvability of the starting vector g^i, which is defined by

$$L_{g^i} L_f^k h_j = \delta_{i,j}\delta_{k,\lambda_i - 1},\ \begin{cases} 1 \le i, j \le p; \\ 0 \le k \le \lambda_i - 1, \end{cases}$$

When g^i is well-defined, $\bar{g}^i$ can be expressed as a linear combination of $g^k, 1 \le k \le p$. We discuss the expression of $\bar{g}^i$ for the case $\lambda_1 = \cdots = \lambda_p$. To simplify the presentation, we assume system (1) is expressed in observable coordinates

$$\begin{aligned} \dot{x} &= f(x), \\ y &= h(x). \end{aligned}$$

In x-coordinates, the equations to solve the starting vector $\bar{g}^k$ can be written as

$$\left[\begin{array}{c} \mathrm{d}h_1 \\ \vdots \\ \mathrm{d}L_f^{\lambda_i-1}h_1 \\ \hline \vdots \\ \hline \mathrm{d}h_p \\ \vdots \\ \mathrm{d}L_f^{\lambda_i-1}h_p \end{array}\right] \left[\begin{array}{c} * \\ \vdots \\ \bar{g}^i_1 \\ \hline \vdots \\ \hline * \\ \vdots \\ \bar{g}^i_p \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ \alpha^i_1 \\ \hline \vdots \\ \hline 0 \\ \vdots \\ \alpha^i_p \end{array}\right],$$

from which we know the starting vector takes the form of $\bar{g}^i = \sum_{k=1}^p \bar{g}^i_k \frac{\partial}{\partial x^k_{\lambda_k}}$. Hence, $\bar{g}^i$ can be solved from

$$\Lambda \begin{bmatrix} \bar{g}^i_1 \\ \vdots \\ \bar{g}^i_p \end{bmatrix} = \begin{bmatrix} \alpha^i_1 \\ \vdots \\ \alpha^i_p \end{bmatrix} = \alpha^i,$$

where

$$\Lambda = \begin{bmatrix} \frac{\partial L_f^{\lambda_1-1}y_1}{\partial x^1_{\lambda_1}} & \cdots & \frac{\partial L_f^{\lambda_1-1}y_1}{\partial x^p_{\lambda_p}} \\ \vdots & \ddots & \vdots \\ \frac{\partial L_f^{\lambda_1-1}y_p}{\partial x^1_{\lambda_1}} & \cdots & \frac{\partial L_f^{\lambda_1-1}y_p}{\partial x^p_{\lambda_p}} \end{bmatrix} = I_p.$$

Therefore, given α^i, $\bar{g}^i$ is uniquely defined in this special case. □

Remark 3.5: Theorem 3.1 is not constructive because

1) Solving (5) may lead to a starting vector $\bar{g}^i$ which depends on α^i_j.
2) α^i_j is non-unique, because (6) and (7) are not sufficient to grantee the uniqueness.

This non-constructive feature of Theorem 3.1 is result from the lack of necessary connection between α and the system dynamics $f(x)$. □

IV. Necessary Conditions on Output Transformations

To address the weakness of the existence conditions in constructing the output transformation, we make use of the Lie bracket condition (8), which implies conditions on α in term of $f(x)$, to establish the necessary conditions on α. These necessary conditions take the form of first order PDEs and therefore relatively easy to check and solve. We first consider the single output (SO) case.

A. Conditions on α: the SO Case

Without loss of generality, we assume the SO system in observable coordinates

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ \vdots \\ \varphi(x) \end{bmatrix}, \qquad (12)$$
$$y = x_1$$

is transformable to an OF with a state transformation $\Phi(x)$ and output transformation $\bar{y} = \Psi(y)$. That is the new system

$$\begin{aligned} \dot{x} &= f(x), \\ \bar{y} &= \Psi(y) = \int \frac{1}{\alpha(y)} \mathrm{d}y \end{aligned} \tag{13}$$

admits an OF. The necessary conditions on α is given in the following proposition.

Proposition 4.1: A locally observable single output system (13) admits an OF with a state transformation $z = \Phi(x)$ and an output transformation $\bar{y} = \Psi(y)$ only if, locally

$$\mathrm{d}L_{\bar{g}} L_f^n y = \frac{n}{\alpha} \frac{\mathrm{d}\alpha}{\mathrm{d}y} \mathrm{d}L_f y \mod \{y\}, \tag{14}$$

where $\bar{g}$ is defined by $L_{\bar{g}} L_f^k y = \alpha \delta_{k,n-1},\ 0 \le k \le n-1$.

Remark 4.2: To show Proposition 4.1, we verify that

$$\begin{aligned} &\mathrm{ad}_{-f}^{i+1} \bar{g} = \alpha \mathrm{ad}_{-f}^{i+1} g - (i+1) L_f(\alpha) \mathrm{ad}_{-f}^{i} g \\ &\quad \mod \{\mathrm{ad}_{-f}^k g, 0 \le k \le i-1\},\ 0 \le i \le n-1, \end{aligned} \tag{15}$$

where g is the starting vector of systems (12), and defined as: for $0 \le k \le n-1$,

$$L_g L_f^k y = \delta_{k,n-1}.$$

From the definitions of $\bar{g}, g$, we can verify

$$\bar{g} = \alpha g.$$

We use induction to show (15). When $k = 1$, the vector field

$$\begin{aligned} \mathrm{ad}_{-f} \bar{g} &= [-f, \alpha g] \\ &= \alpha \mathrm{ad}_{-f} g - L_f(\alpha) g. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \mathrm{ad}_{-f}^2 \bar{g} &= [-f, \alpha \mathrm{ad}_{-f} g - L_f(\alpha) g] \\ &= \alpha \mathrm{ad}_{-f}^2 g - 2 L_f(\alpha) \mathrm{ad}_{-f} g + L_f^2(\alpha) g \\ &= \alpha \mathrm{ad}_{-f}^2 g - 2 L_f(\alpha) \mathrm{ad}_{-f} g \mod \{g\} \end{aligned}$$

Assuming for $3 \le i \le n-2$,

$$\begin{aligned} \mathrm{ad}_{-f}^i \bar{g} =& \alpha \mathrm{ad}_{-f}^i g - i L_f(\alpha) \mathrm{ad}_{-f}^{i-1} g \\ & \mod \{\mathrm{ad}_{-f}^k g, 0 \le k \le i-2\}, \end{aligned}$$

we compute $\mathrm{ad}_{-f}^{i+1} \bar{g}$

$$\begin{aligned} \mathrm{ad}_{-f}^{i+1} \bar{g} =& [-f, \alpha \mathrm{ad}_{-f}^i g - i L_f(\alpha) \mathrm{ad}_{-f}^{i-1} g] \\ & [-f, \mod \{\mathrm{ad}_{-f}^k g, 0 \le k \le i-2\}] \\ =& \alpha \mathrm{ad}_{-f}^{i+1} g - (i+1) L_f(\alpha) \mathrm{ad}_{-f}^i g \\ & \mod \{\mathrm{ad}_{-f}^k g, 0 \le k \le i-1\}. \end{aligned}$$

We therefore verify that (15) holds. □

The proof of Proposition 4.1 is given as follows.

Proof: Assume system (13) is transformed into an OF

$$\begin{aligned} \dot{z} &= Az + \gamma(\bar{y}), \\ \bar{y} &= Cz, \end{aligned}$$

where $z = (z_1, \dots, z_n)^T$, and A, C are in Brunovsky Form. This implies that the following equations hold

$$\frac{\partial W}{\partial z_{j-1}} = \mathrm{ad}_{-f} \frac{\partial W}{\partial z_j}, \quad 2 \le j \le n, \tag{16a}$$

$$\frac{\partial W}{\partial z} \frac{\partial}{\partial z_1} (Az + \gamma(\bar{y})) = \mathrm{ad}_{-f} \frac{\partial W}{\partial z_1},, \tag{16b}$$

where $W = \Phi^{-1}(z)$, and $\partial W / \partial z_n$ is the starting vector $\bar{g}$. In z-coordinates, (16b) is rewritten as

$$\frac{\partial}{\partial z_1} (Az + \gamma(\bar{y})) = \mathrm{ad}_{-f}^n \bar{g}. \tag{17}$$

The left hand side is

$$\frac{\partial \gamma \circ \Psi(y)}{\partial y} \frac{\partial y}{\partial \bar{y}} = \frac{\partial \gamma \circ \Psi(y)}{\partial y} \alpha.$$

The Lie derivatives of y along the vector fields in both sides of (17) are

$$\begin{aligned} \rho(y) =& \left\langle \mathrm{d}y, \alpha \mathrm{ad}_{-f}^n g - n L_f(\alpha) \mathrm{ad}_{-f}^{n-1} g \right\rangle \\ =& \alpha L_g L_f^n y - n L_f(\alpha), \end{aligned} \tag{18}$$

where

$$\rho(y) = \left\langle \mathrm{d}y, \alpha \frac{\partial \gamma \circ \Psi(y)}{\partial y} \right\rangle.$$

Here we apply [8, Lem. 4.1.2] to obtain

$$\left\langle \mathrm{d}y, \mathrm{ad}_{-f}^n g \right\rangle = L_g L_f^n y.$$

Taking the differential of (18), we have

$$\alpha \mathrm{d}L_g L_f^n y = n \frac{\mathrm{d}\alpha}{\mathrm{d}y} \mathrm{d}L_f h \mod \{y\}$$

We therefore show that the condition (14) is necessary. ■

B. Conditions on α: the MO Case

A locally observable system (1) does not admit an OF because of the following two reasons:

1) The starting vector $g^i, 1 \le i \le p$ cannot be solved.
2) Given g^i well-defined, the Lie bracket condition does not hold.

The introduction of an output transformation might lead to the solvability of the start vectors or ensure the Lie bracket condition satisfied. To simplify the problem, we make the following assumption.

Assumption 4.3: Given a locally observable system (1), the starting vectors $g^i, 1 \le i \le p$ are well-defined.

Proposition 4.4: Given Assumption 4.3, a locally observable multi-output system (1) admits an OF with a state transformation $z = \Phi(\zeta)$ and an output transformation $\bar{y} = \Psi(y)$ only if, locally

$$\begin{aligned} \sum_{k=1}^{p} \alpha_k^i \mathrm{d}L_{g^i} L_f^{\lambda_i} y_i =& \lambda_i \sum_{k=1}^{p} \sum_{r=1}^{p} \frac{\partial \alpha_k^i}{\partial y_r} \mathrm{d}L_f y_r \\ & \mod \{\mathrm{d}y\}, \quad 1 \le i \le p, \end{aligned} \tag{19}$$

where g^i is defined by

$$L_{g^i} L_f^k h_j = \delta_{i,j} \delta_{k,\lambda_i - 1}, \begin{cases} 1 \le i, j \le p; \\ 0 \le k \le \lambda_i - 1. \end{cases}$$

By Assumption 4.3, g^i are well-defined. Hence, the starting vectors $\bar{g}^i$ can be expressed as

$$\bar{g}^i = \sum_{k=1}^{p} \alpha_k^i g^k.$$

Remark 4.5: Similarly to Remark 4.2, we can verify that

$$\begin{aligned} \mathrm{ad}_{-f}^{l+1}\bar{g}^i = &\sum_{k=1}^{p}(\alpha_k^i \mathrm{ad}_{-f}^{l+1} g^i - (l+1)L_f(\alpha_k^i)\mathrm{ad}_{-f}^{l} g^i) \\ &\mod \{\mathrm{ad}_{-f}^{k} g^i, 0 \le k \le l-1\}, \quad 0 \le l \le \lambda_i - 1. \end{aligned} \tag{20}$$

When $l = 1$, we have

$$\begin{aligned} \mathrm{ad}_{-f}\bar{g}^i =& [-f, \sum_{k=1}^{p}\alpha_k^i g^k] \\ =& \sum_{k=1}^{p}(\alpha_k^i \mathrm{ad}_{-f} g^i - L_f(\alpha_k^i) g^i). \end{aligned}$$

Assuming

$$\begin{aligned} \mathrm{ad}_{-f}^{l}\bar{g}^i =& \sum_{k=1}^{p}(\alpha_k^i \mathrm{ad}_{-f}^{l} g^i - lL_f(\alpha_k^i)\mathrm{ad}_{-f}^{l-1} g^i) \\ &\mod \{\mathrm{ad}_{-f}^{k} g^i, 0 \le k \le l-2\}, \end{aligned}$$

we verify that (20) holds for the $l + 1$ case. □

We are ready to give the proof of Proposition 4.4.

Proof: The proof follows the same procedure as the SO case. Since system (1) is transformable to an OF by a change of coordinates $z = \Phi(\zeta)$ and an output transformation $\bar{y} = \Psi(y)$, we know the following conditions hold

$$\frac{\partial W}{\partial z_{j-1}^i} = \mathrm{ad}_{-f}\frac{\partial W}{\partial z_j^i}, \quad 2 \le j \le \lambda_i, \tag{21a}$$

$$\frac{\partial W}{\partial z}\frac{\partial}{\partial z_1^i}(Az + \gamma(\bar{y})) = \mathrm{ad}_{-f}\frac{\partial W}{\partial z_1^i}, \tag{21b}$$

where $W = \Phi^{-1}(z)$, and $\partial W/\partial z_{\lambda_i}^i$ is the starting vector $\bar{g}^i$. This fact has been shown in [25]. According to Remark 4.5, (21b) in z-coordinates is written as

$$\begin{aligned} \frac{\partial}{\partial z_1^i}(Az + \gamma(\bar{y})) =& \sum_{k=1}^{p}(\alpha_k^i \mathrm{ad}_{-f}^{\lambda_i} g^i - \lambda_i L_f(\alpha_k^i)\mathrm{ad}_{-f}^{\lambda_i - 1} g^i) \\ &\mod \{\mathrm{ad}_{-f}^{k} g^i, 0 \le k \le \lambda_i - 2\}, \end{aligned}$$

which gives

$$\begin{aligned} \frac{\partial \gamma \circ \Psi}{\partial y}\frac{\partial y}{\partial z_1^i} =& \sum_{k=1}^{p}(\alpha_k^i \mathrm{ad}_{-f}^{\lambda_i} g^i - \lambda_i L_f(\alpha_k^i)\mathrm{ad}_{-f}^{\lambda_i - 1} g^i) \\ &\mod \{\mathrm{ad}_{-f}^{k} g^i, 0 \le k \le \lambda_i - 2\}. \end{aligned} \tag{22}$$

The Lie derivatives of y_i along the vector fields on both sides of (22) are

$$\begin{aligned} \rho(y) =& \sum_{k=1}^{p}(\alpha_k^i L_{\mathrm{ad}_{-f}^{\lambda_i} g^i} y_i - \lambda_i L_f(\alpha_k^i) L_{\mathrm{ad}_{-f}^{\lambda_i - 1} g^i} y_i) \\ =& \sum_{k=1}^{p}(\alpha_k^i L_{g^i} L_f^{\lambda_i} y_i - \lambda_i L_f(\alpha_k^i)), \end{aligned}$$

where we abuse the notation ρ. Taking the differential of above equation, we have

$$\sum_{k=1}^{p}\alpha_k^i \mathrm{d}L_{g^i} L_f^{\lambda_i} y_i = \lambda_i \sum_{k=1}^{p}\sum_{r=1}^{p}\frac{\partial \alpha_k^i}{\partial y_r}\mathrm{d}L_f y_r \mod \{\mathrm{d}y\}.$$

We therefore show the necessary conditions on α^i. ■

V. A Perspective System Example

A perspective dynamic system with three states and two outputs, derived assuming a calibrated pinhole camera and observations of feature points on a rigid object, can be written as

$$\dot{\zeta} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\zeta + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad y = \begin{bmatrix} \zeta_1/\zeta_3 & \zeta_2/\zeta_3 \end{bmatrix}^T, \tag{23}$$

where $a_{ij}, b_i, 1 \le i, j \le 3$ are constant and $\zeta_3 > 0$ [10]. System (23) is observable with observability indices $\lambda_1 = 2, \lambda_2 = 1$ provided $(b_1 - b_3\zeta_1/\zeta_3)^2 + (b_2 - b_3\zeta_2/\zeta_3)^2 \neq 0$. The output y making system (23) lose observability is called the *focus of expansion* [9]. Since Condition 2) in Theorem 3.1 is violated for the case $i = 1$, g^1 does not exist. Proposition 4.4 is not directly applicable. We first need to solve an output transformation $\bar{y} = (\psi_1(y), \psi_2(y))^T$ such that g^1 is solvable. Condition 2) in Theorem 3.1 yields a PDE

$$\frac{\partial \psi_2}{\partial y_1}(b_1 - b_3 y_1) + \frac{\partial \psi_2}{\partial y_2}(b_2 - b_3 y_2) = 0. \tag{24}$$

Solving (24) gives a solution

$$\psi_2(y_1, y_2) = \frac{b_3 y_2 - b_2}{b_3(b_3 y_1 - b_1)}.$$

With new output $y = (y_1, \psi_2)$, the observable form of system (23) takes the expression

$$\dot{x} = \begin{bmatrix} x_2^1 \\ \varphi_1(x) \\ \varphi_2(x) \end{bmatrix}, \quad y = \begin{bmatrix} x_1^1 \\ x_1^2 \end{bmatrix},$$

where the notation y is reused. Given $L_f^2 y_1 = \varphi_1, L_f y_2 = \varphi_2$, applying Proposition 4.4 leads to the following conditions on $\alpha_k^1, \alpha_k^2, k = \{1, 2\}$

$$\begin{aligned} &(\alpha_1^1 + \alpha_2^1)\frac{\partial L_{g^1}\varphi_1}{\partial x_2^1}\mathrm{d}x_2^1 \\ &= 2(\frac{\partial \alpha_1^1}{\partial y_1}\mathrm{d}x_2^1 + \frac{\partial \alpha_1^1}{\partial y_2}\mathrm{d}\varphi_2 + \frac{\partial \alpha_2^1}{\partial y_1}\mathrm{d}x_2^1 + \frac{\partial \alpha_2^1}{\partial y_2}\mathrm{d}\varphi_2) \mod \{\mathrm{d}y\}, \\ &(\alpha_1^2 + \alpha_2^2)\mathrm{d}L_{g^2}\varphi_2 \\ &= 2(\frac{\partial \alpha_1^2}{\partial y_1}\mathrm{d}x_2^1 + \frac{\partial \alpha_1^2}{\partial y_2}\mathrm{d}\varphi_2 + \frac{\partial \alpha_2^2}{\partial y_1}\mathrm{d}x_2^1 + \frac{\partial \alpha_2^2}{\partial y_2}\mathrm{d}\varphi_2) \mod \{\mathrm{d}y\}, \end{aligned}$$

Since φ_2 depends on y only, the above conditions are reduced to

$$(\alpha_1^1 + \alpha_2^1)\frac{\partial L_{g^1}\varphi_1}{\partial x_2^1} = 2(\frac{\partial \alpha_1^1}{\partial y_1} + \frac{\partial \alpha_2^1}{\partial y_1}), \quad \frac{\partial \alpha^2}{\partial y_1} = 0.$$

Further computation gives $g^1 = \partial/\partial x_2^1$, and $L_{g^1}\varphi_1 = 4b_3/(y_1b_3 - b_1)$. Denoting $\bar{\alpha}^1 = \alpha_1^1 + \alpha_2^1$, we have a PDE

$$\frac{2b_3}{(y_1b_3 - b_1)} = \frac{1}{\bar{\alpha}^1}\frac{\partial\bar{\alpha}^1}{\partial y_1}.$$

Solving the PDE gives

$$\bar{\alpha}^1 = \frac{C}{(y_1b_3 - b_1)^2}, \quad C \in \mathbb{R}\backslash\{0\}.$$

From simplicity, we take $\alpha_2^1 = \alpha_1^2 = 0, \alpha_2^2 = 1$ s.t. α satisfies (6) and (7). Solving (9) gives the output transformation $\psi_1 = 1/(b_1 - b_3y_1)$. Work [4] has shown that system (23) admits an OF (3) with the output transformation $\bar{y} = (\psi_1, \psi_2)^T$ and a local diffeomorphism $z = \Phi(\zeta)$.

VI. Conclusion

This paper discussed the existence conditions under which an unforced multi-output nonlinear system can be transformed into a multi-output observer form by an output transformation and a change of state coordinates. Different from the existing work without considering output transformation, Lie bracket conditions in the output space are imposed to ensure the solvability of output transformation candidates. Given an output transformation, verifying these conditions can reveal if the unforced system is transformable to the observer form. Existence conditions however are not constructive in the sense that the output transformation can not be solved explicitly. Necessary conditions on the output transformation candidates are given for the single output and multi-output nonlinear systems. These necessary conditions are in the form of first order PDEs and relatively easy to verify and solve, thus potentially useful to solve the output transformation.

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