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TR2011-007   December 2010

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IEEE Communications Letters

The Gamma Variate with Random Shape Parameter and Some Applications

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***Abstract*—This letter provides a new outlook at the Gamma distribution and its involvement in the performance analysis of digital communications over wireless fading channels. Whereas the Gamma distribution is usually regarded as a two-parameter distribution with a scale and shape factors that are deterministic parameters, we investigate the implications and insight that can be drawn from considering a Gamma variate with a random shape factor. Applications of such a distribution in the context of information transmission over a wireless fading channel are provided and novel average symbol error probability (SEP) expressions for single and multi-channel reception are derived involving the Laplace transform of the new distribution. We analytically prove and verify via numerical simulations that the average SEP results induced by the random shape parameter are lower bounded by those obtained using a deterministic fading severity whose value equals the expected value of the random shape parameter.**

***Index Terms*—Gamma distribution, digital communication systems, fading/shadowing channel models, receiver performance analysis.**

I. Introduction

The Gamma variate which belongs to the family of continuous distributions, has long been associated with the modeling of wireless fading channels and as such has played an important role in the performance analysis of digital communication systems over fading channels. As a notable example, small-scale fading of the wireless channel gain is often modeled by the versatile Nakagami-m distribution which has been shown to fit experimental data [1] and whose corresponding power statistics are Gamma distributed [2]. The appeal of the Gamma distribution owes as much to its simplicity as to its mathematical tractability compared to other distributions which may not always lend themselves to further analytical derivations.

A Gamma-distributed random variable (RV) β is denoted as $\beta \sim \Upsilon(k, \Omega)$, with $k > 0$ being the shape factor and $\Omega > 0$ the scale parameter. In the context of information transmission over wireless fading channels, the scale parameter Ω relates to the average fading power whereas the fading/shadowing severity is captured by the shape parameter k. In many practical scenarios, notably those involving user/scatterer mobility where the fading environment is constantly changing, it may not be possible to determine the fading severity a priori. Unsurprisingly, recent measurement campaigns conducted in the urban area of Valencia in Spain [3] prove that the Nakagami-m parameter, hence the shape factor of the underlying Gamma random variable used to model the fading channel, is inherently variable even when observed within the confines of the same urban environment. A normalized histogram of the m parameter at 900MHz band [3, Fig. 3] reveals that m actually ranges from $m_{\min} = 0.5$ to $m_{\max} = 3.5$ with an average value of $\mu_m = 1.56$ and a standard deviation of $\sigma_m = 0.34$. Hence the fading severity does indeed vary from “severely fading” to “lightly fading”. In general, the fading severity is influenced by a number of parameters such as the environment type, the operating frequency, the transmitter and receiver antenna heights, polarizations and separation distance as well as the relative position of the scatterers. In practice, all these parameters are subject to change even assuming a low mobility or static fading environment. Therefore, instead of treating the fading severity index as deterministic, we propose to model it as a RV and evaluate the Laplace transform (LT) of the ensuing Gamma variate when its shape parameter is drawn from some distribution that is known a priori, thereby revealing interesting insights on the Gamma distribution when leveraged from this new angle.

The remainder of this letter is as follows. In Section II, we present the Gamma distribution with random shape parameter and determine its Laplace transform. In Section III, we provide applications of the new distribution to the performance analysis and average SEP evaluation of digital modulations over fading channels with and without diversity reception. Section IV provides numerical results along with an associated technical discussion and finally concluding remarks are presented in Section V.

II. Gamma Distribution with Random Shape Parameter

Let $\beta \sim \Upsilon(k, \Omega)$. Hence, the probability density function (PDF) of β is given by

$$f_\beta(x) = \frac{e^{-\frac{x}{\Omega}} x^{k-1}}{\Omega^k \Gamma(k)}, \quad k \geq 1 \quad \text{and} \quad x \geq 0, \tag{1}$$

where $\Gamma(n) = \int_0^\infty e^{-u} u^{n-1} du$ is the standard Gamma function. The LT of the PDF (1), $\mathcal{L}_\beta(s) = E[e^{-s\beta}]$, can be expressed as

$$\mathcal{L}_\beta(s) = (1 + s\Omega)^{-k}, \quad \Re\{s\} \geq 0. \tag{2}$$

Note that (1) and (2) are valid when Ω and k are deterministic parameters.

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When the shape parameter k is drawn from a probability distribution, (1) and (2) represent the conditional PDF and conditional LT, conditioned on $K = k$, where K is a RV representing the shape parameter. Focusing on (2), we write

$$\begin{aligned}\mathcal{L}_{\beta}(s) &= E\left[e^{-s\beta}\right] = E\left[E\left\{e^{-s\beta} \mid K = k\right\}\right] \\ &= E\left[(1+s\Omega)^{-K}\right] = E\left[e^{-\log(1+s\Omega)K}\right] \\ &= \mathcal{L}_K\left(\log\left(1+s\Omega\right)\right). \qquad (3)\end{aligned}$$

Note that (3) is valid when the RV K is continuous or discrete.

The expected value of β can be obtained from (3) as

$$\begin{aligned}E[\beta] &= -\frac{d}{ds}\mathcal{L}_{\beta}(s)\Big|_{s=0} \\ &= \frac{\Omega}{1+s\Omega}E\left[Ke^{-\log(1+s\Omega)K}\right]\Big|_{s=0} \\ &= \Omega E[K]. \qquad (4)\end{aligned}$$

Note that (4) can alternatively be obtained as $E[\beta] = E\left[E\left\{\beta \mid K = k\right\}\right] = E[\Omega k] = \Omega E[K]$.

Since $\alpha = \log\left(1+s\Omega\right) > 0$, making use of Jensen's inequality implies that $E[e^{-\alpha K}] \geq e^{-\alpha E[K]}$. That is, a lower bound on (3) is obtained as follows

$$\mathcal{L}_{\beta}(s) \geq (1+s\Omega)^{-E[K]}. \qquad (5)$$

Note that (5) becomes an equality when $K = k$ with probability one.

We now extend (3) and (5) to the case of multiple Gamma RVs with *arbitrarily correlated* shape parameters. We consider N RVs $\beta_1, \ldots, \beta_N$. Let K_n denote the shape parameter RV associated with β_n. We assume that conditioned on $K_1, \ldots, K_N$ $\beta_1, \ldots, \beta_N$ are independent. That is

$$f_{\beta_1,\ldots,\beta_N|K_1,\ldots,K_N}\left(x_1,\ldots,x_N\right) = \prod_{n=1}^{N} f_{\beta_n|K_n}\left(x_n\right). \qquad (6)$$

The joint LT of $\beta_1, \ldots, \beta_N$ is obtained as

$$\begin{aligned}&\mathcal{L}_{\beta_1,\ldots,\beta_N}\left(s_1,\ldots,s_N\right) = E\left[e^{-\sum_{n=1}^{N} s_n\beta_n}\right] \\ &= E\left[E\left[e^{-\sum_{n=1}^{N} s_n\beta_n}\Big| K_1,\ldots,K_N\right]\right] \\ &= E\left[\prod_{n=1}^{N} E\left[e^{-s_n\beta_n}\Big| K_n\right]\right] \\ &= E\left[\prod_{n=1}^{N}\left(1+s_n\Omega_n\right)^{-K_n}\right] \\ &= \mathcal{L}_{K_1,\ldots,K_N}\left(\log\left(1+s_1\Omega_1\right),\ldots,\log\left(1+s_N\Omega_N\right)\right) \qquad (7)\end{aligned}$$

whereas Jensen's inequality yields

$$\begin{aligned}&\mathcal{L}_{K_1,\ldots,K_N}\left(\log\left(1+s_1\Omega_1\right),\ldots,\log\left(1+s_N\Omega_N\right)\right) \\ &\geq \prod_{n=1}^{N}\left(1+s_n\Omega_n\right)^{-E[K_n]}. \qquad (8)\end{aligned}$$

As will be shown in Section III, expressions in (3) and (5), as well as their multivariate counterparts in (7) and (8), have direct applications to the performance analysis of linear modulation schemes over wireless fading channels.

III. Applications

The fading severity index–also known as the scintillating index or coefficient of variation (CV) [4][1]–of the single-variate Gamma distribution with random shape parameter is computed as

$$\mathrm{CV} = \sqrt{\frac{E[\beta^2]}{E^2[\beta]} - 1} = \frac{\sqrt{\mathsf{Var}\left(K\right) + \overline{K}}}{\overline{K}}, \qquad (9)$$

where $\mathsf{Var}\left(K\right)$ denotes the variance of K. Here, it is worthwhile noticing that the CV depends on the first two moments of the shape factor distribution rather than its actual value.

Now, consider communication over a single-input single-output (SISO) flat fading channel. With E_s the average symbol energy and N_0 the one-sided power-spectral density of the additive white Gaussian random process, the instantaneous received signal-to-noise ratio (SNR) is $\gamma = (E_s/N_0)\beta$, where β denotes the instantaneous fading power. When β is a Gamma RV with random shape parameter, the average received SNR is $\overline{\gamma} = E[\gamma] = (E_s/N_0)E[\beta] = (E_s/N_0)\Omega\overline{K}$, where $\overline{K} = E[K]$ is the average fading severity. That is, we can express the scale parameter Ω as

$$\Omega = \overline{\gamma}\frac{N_0}{E_s\overline{K}}. \qquad (10)$$

Assuming M-PSK modulation, the average SEP is [5]

$$\begin{aligned}\overline{P}_{s,M-PSK} &= \frac{1}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \mathcal{L}_{\gamma}\left(\frac{\sin^2\frac{\pi}{M}}{\sin^2\theta}\right)d\theta \\ &= \frac{1}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \mathcal{L}_{\beta}\left(\frac{E_s}{N_0}\frac{\sin^2\frac{\pi}{M}}{\sin^2\theta}\right)d\theta \\ &= \frac{1}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \mathcal{L}_K\left(\log\left(1+\frac{\overline{\gamma}}{\overline{K}}\frac{\sin^2\frac{\pi}{M}}{\sin^2\theta}\right)\right)d\theta. \qquad (11)\end{aligned}$$

Upon using (5) in (11), we have

$$\overline{P}_{s,M-PSK} \geq \int_0^{\frac{(M-1)\pi}{M}} \frac{d\theta}{\pi}\left(\frac{1}{1+\frac{\overline{\gamma}}{\overline{K}}\frac{\sin^2\frac{\pi}{M}}{\sin^2\theta}}\right)^{\overline{K}}, \qquad (12)$$

proving that the average SEP induced by the random shape parameter is in fact lower bounded by the one obtained with a deterministic fading severity index whose value equals the expected value of the random fading severity index. This is not only an intuitive result but also one that is valid for a wide range of fading severity index distributions.

We can also consider the case with L antennas at the receiver with maximal ratio combining (MRC). Here, the instantaneous received SNR on the lth branch is denoted by γ_l, whereas the SNR at the output of the MRC receiver is $\gamma = \sum_{l=1}^{L}\gamma_l$. We denote the scale and shape parameters on the lth branch by Ω_l and K_l, respectively. Similar to (10), we have

$$\Omega_l = \overline{\gamma}_l\frac{N_0}{E_s\overline{K}_l}, \qquad (13)$$

[1]Note that the CV is related to another fading severity measure which is the amount of fading (AF) defined as AF := CV^2.

where $\overline{\gamma}_l = (E_s/N_0)E[\beta_l]$ is the average received SNR and $\overline{K}_l = E[K_l]$ is the average fading severity on the lth branch, respectively.

Assuming arbitrarily correlated $\beta_1, \ldots, \beta_L$, as in (6), we have $\mathcal{L}_\gamma(s) = \mathcal{L}_{\gamma_1,\ldots,\gamma_L}(s,\ldots,s)$, and an expression similar to (11) and (12) can be obtained as

$$\overline{P}_{s,M-PSK} = \int_0^{\frac{(M-1)\pi}{M}} \frac{d\theta}{\pi} \mathcal{L}_{K_1,\ldots,K_L}\Bigg(\log\left(1+\frac{\overline{\gamma}_1}{\overline{K}_1}\frac{\sin^2\frac{\pi}{M}}{\sin^2\theta}\right),\ldots,\log\left(1+\frac{\overline{\gamma}_L}{\overline{K}_L}\frac{\sin^2\frac{\pi}{M}}{\sin^2\theta}\right)\Bigg) \quad (14)$$

$$\geq \int_0^{\frac{(M-1)\pi}{M}} \frac{d\theta}{\pi} \prod_{l=1}^{L} \left(\frac{1}{1+\frac{\overline{\gamma}_l}{\overline{K}_l}\frac{\sin^2\frac{\pi}{M}}{\sin^2\theta}}\right)^{\overline{K}_l}. \quad (15)$$

Expressions similar to (11), (12), (14) and (15) can also be obtained for other modulation formats such as M-QAM. We note that (11) and (14) are valid for arbitrary distributions of the fading severity index K, and we only need the LT of K to compute the average error rates.

IV. Numerical Results

In this section, we provide numerical results to illustrate the analytical ones obtained in the previous two sections. Fig. 1 shows the average SEP of M-PSK modulation with $M \in \{2, 8\}$ on Gamma fading channels. Both deterministic and random fading severity indexes are considered. With deterministic fading severity index, we consider $k = 1$ and $k = 5$, whereas with random fading severity index we assume that K is uniformly distributed over the interval $(1,\ 5]$. From Fig. 1, we conclude that $k = 1$ leads to a first-order diversity performance, whereas $k = 5$ predicts a fifth-order diversity performance. On the other hand, with a uniformly distributed random fading severity index, the diversity order of average SEP is conservatively estimated to be within that induced by the deterministic fading severity indexes. Fig. 1 further upholds the lower bound obtained using a deterministic fading severity whose value equals the expected value of the random shape parameter. Fig. 2 illustrates the average bit error rate (BER) performance of 16-QAM on measured Nakagami-m fading channels [3] where a discrete approximation for the distribution of the Nakagami parameter m is shown in the upper part of Fig. 2. In addition to the middle curve representing the BER of 16-QAM averaged over both the fading distribution as well as the measured distribution of the m parameter, a lower and upper bounds on the average BER performance are also depicted pertaining to the average BER results of 16-QAM with deterministic m values corresponding to $m = 0.625$ and $m = 3.375$, respectively.

V. Conclusion

In this letter, we introduce the Gamma distribution with random shape factor as a new model for analyzing the performance of digital communications over fading channels. It is noteworthy to mention that the random shape factor idea investigated in this letter can easily be exported to a Rician fading channel model with random line-of-sight (LOS) component, in which case analog results for the CV, the average SEP and its lower bound can also be derived.

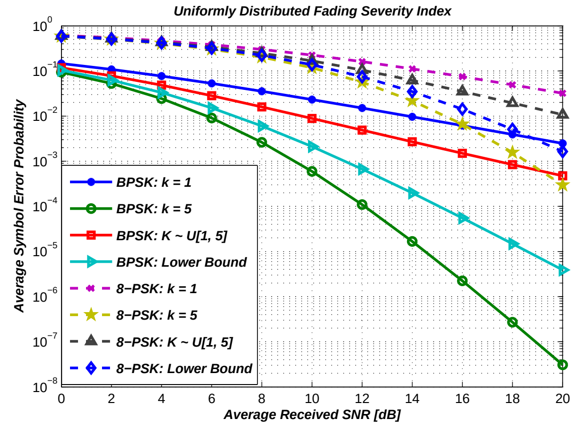

Fig. 1. Average SEP of BPSK and 8-PSK modulations on Gamma fading channels with uniform fading severity index.

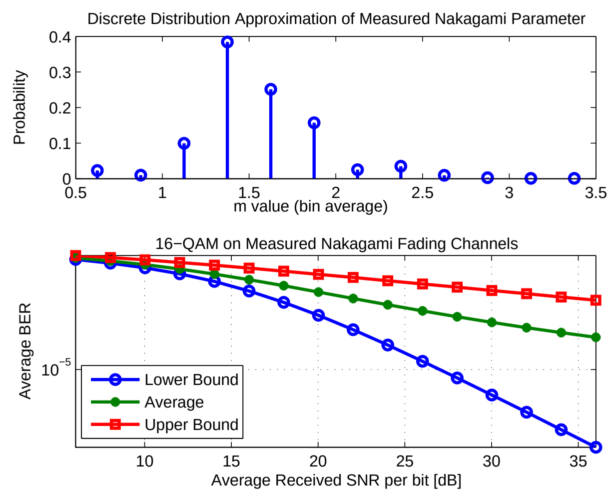

Fig. 2. Average BER of 16-QAM on measured Nakagami-m fading channels.

Acknowledgment

The authors would like to thank Dr. L. Rubio for providing us with the data traces of the measurement campaign reported in [3].

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