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Chen, Z.; Sahinoglu, Z.; Li, H.

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Fast Frequency and Phase Estimation in Three Phase Power Systems

Zhu Chen, *Student Member, IEEE,* Zafer Sahinoglu, *Senior Member, IEEE,* Hongbin Li, *Senior Member, IEEE*

***Abstract*—In this paper, we investigate the fundamental frequency and phase estimation problem in a balanced three-phase power system with harmonic distortion. An estimation algorithm that can rapidly track the fast-changing frequency and phase by using quarter cycle samples are proposed. Specifically, the data model in the three-phase power system is first converted to the noise-corrupted single phase harmonic signal model by using the Clarke transformation. A new weighted least squares (WLS) parameter estimator, which refines the initial estimates from the standard estimation techniques, is computed by utilizing the harmonic structure of the signal. Since the initial estimates become unreliable with limited samples, we proposed an iterative algorithm to polish the initial estimates for WLS. Numerical results show that the proposed estimator outperforms the conventional estimators, especially in a data-limited case.**

***Index Terms*—Phasor extraction, fundamental frequency estimation, phase estimation, harmonic structure, multiple signal classification, weighted least squares.**

I. Introduction

Power quality is becoming a major concern in order to ensure the reliability of power delivery and maintain the voltage characteristics within certain limits [1]. The main characteristics of the voltage waveforms to be measured are the magnitude, fundamental frequency and phase. In power system, the typical use of frequency and phase estimation is for protection against loss of synchronism, under-frequency relaying and power system stabilization. However, the frequency and phase of a distribution network can extremely quickly vary during transient events, and it can be very difficult to track the frequency and phase with enough accuracy [2]. Thus, fast and accurate frequency and phase estimation in the presence of harmonic distortion and noise is a challenging problem that has attracted much attention.

Various techniques have been developed to extract the phasor in power system. Zero-crossing detection algorithm is widely employed and easy to implement, but it requires the appropriate filtering and large time windows to obtain accurate measurements [3]. The filtering technique by using DFT and recursive DFT algorithm needs a time window which has to be multiple of the fundamental period in order to obtain accurate results. A three-phase phase-locked loop (3PLL) is a structure that is widely used to estimate and track the frequency of three-phase (3PH) power signals. Numerous architectures have been proposed to construct a 3PLL mechanism [4]–[6]. However, in all cases, the 3PLL shows a transient response above one cycle. A fast frequency estimation method is present in [7], by which the response time is around half cycle of the grid voltage, however, this method requires a filter to eliminate the harmonics distortion and it does not consider the phase delay or measurement delay generated by the filter.

Z. Chen and H. Li are with the Department of Electrical and Computer Engineering, Stevens Institute of Technology, Hoboken, NJ 07030 USA (e-mail: zchen2@stevens.edu; Hongbin.Li@stevens.edu).

Z. Sahinoglu is with Mitsubishi Electric Research Laboratories (MERL), Cambridge, MA 02139 USA (e-mail: zafer@merl.com).

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Despite the fact that there are many algorithms used to estimate the frequency [8]–[12], most of them are based on the measurement of a single phase of the system, thus these algorithms exhibit poor behavior when the tracked phase suffers a dip or a transient. The Clarke's α, β transformation is widely used to convert 3PH quantities to a complex quantity in a single-phase system which has a classic harmonic signal model. The parameter estimator based on this single-phase system is more robustness due to the utilization of the information from all the three-phase voltage. In [13], [14] the authors consider the overall information from 3PH voltage when they design the frequency estimator, but they still need the adaptive FIR filter which requires multiple fundamental period to eliminate the harmonics. The multiple signal classification (MUSIC) [15] estimation criterion and estimation of signal parameters via rotational invariance techniques (ESPRIT) [16] can be used for high resolution estimation of the fundamental frequency and phase of the harmonic signal. However, both of them ignore the useful information from the harmonic distortion. In [17], [18], the developed MUSIC parameter estimations utilize the harmonic structure and can achieve a great improvement. Especially, the estimator present in [19] using a Markov-like WLS (MWLS) to refine the initial estimation from MUSIC provides a good estimation performance which is very close to the CRB with large observations or high SNR. Moreover, the MWLS is also able to significantly reduce the computational complexity compared with the optimal nonlinear least-squares method.

In this paper, we design the fundamental frequency and phase estimators by considering the data-limited constraint: there are quarter cycle samples are available for the system. These estimators have great significance in practice due to their accuracy and fast response around quarter cycle. To be specific, we first apply the Clarke transformation to convert the 3PH system to a single-phase system with a harmonic signal model. By extending the WLS technique [19] to our signal model, the new frequency and phase estimators are computed. Since the advantage of the new WLS parameter estimators are lost due to the data-limited constraint, in terms of the characteristics of the harmonic component in 3PH power system, we then propose an improved WLS (IWLS) scheme to increase the estimation accuracy.

II. Problem Formulation

A. Signal Model in 3PH Power System

In a balanced three-phase power system with $K-1$ harmonic distortion, the observed signal can be modeled as:

$$\begin{aligned} v_a(n) &= V_1\cos(\omega_0 n+\phi)+\sum_{k=2}^{K}V_k\cos k(\omega_0 n+\phi)+\epsilon_a(n)\\ v_b(n) &= V_1\cos(\omega_0 n+\phi_b)+\sum_{k=2}^{K}V_k\cos k(\omega_0 n+\phi_b)+\epsilon_b(n)\\ v_c(n) &= V_1\cos(\omega_0 n+\phi_c)+\sum_{k=2}^{K}V_k\cos k(\omega_0 n+\phi_c)+\epsilon_c(n) \end{aligned} \tag{1}$$

where $\phi_b=\phi-\frac{2}{3}\pi$, $\phi_c=\phi+\frac{2}{3}\pi$, n is the time instant and $n=1,2,...,N$, ω_0 is the fundamental frequency with $0<|\omega_0|<\pi$ and the phase $\phi\in[0,2\pi)$, $V_k>0$ is the magnitude of the kth harmonic. The noise vector $\boldsymbol{\epsilon}_{abc}(n)=\begin{bmatrix}\epsilon_a(n) & \epsilon_b(n) & \epsilon_c(n)\end{bmatrix}^T$ is assumed to be a zero mean Gaussian random vector with covariance matrix $\sigma^2\mathbf{I}$.

Give the observations $\{v_i(n)\}_{n=1}^N$ with $i=a,b,c$, the problem of interest is to estimate the unknown parameters: the fundamental frequency ω_0 and the phase ϕ with only quarter cycle samples.

B. Harmonic Signal Model in Single-phase System

Applying the Clarke transformation to (1), we obtain the corresponding signal in $\alpha\beta$ stationary reference frame as

$$\begin{bmatrix} v_\alpha(n)\\ v_\beta(n)\end{bmatrix}=\mathbf{T}\begin{bmatrix} v_a(n)\\ v_b(n)\\ v_c(n)\end{bmatrix} \tag{2}$$

where the transform matrix is given by:

$$\mathbf{T}=\frac{2}{3}\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2}\\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2}\end{bmatrix}. \tag{3}$$

The complex voltage obtained by $v(n)=v_\alpha(n)+jv_\beta(n)$ can be expressed in terms of a harmonic signal model

$$v(n)=\sum_{k=1}^{K}A_k e^{jl_k(\omega_0 n+\phi)}+\epsilon(n), n=1,2,...,N \tag{4}$$

where $l_k=\frac{(-1)^{k-1}(6k-3)+1}{4}$, $A_k=V_{|l_k|}$ and $\epsilon(n)\sim\mathcal{CW}(0,\frac{4}{3}\sigma^2)$. It can be seen that there are totally K harmonics involved in this model and the order of the kth harmonic is l_k. This signal model synthesized from the three single-phase signals is more reliable and robust whenever an arbitrary phase suffers dips, transients, or small interruptions.

Let $\mathbf{v}=\begin{bmatrix}v(1) & \ldots & v(N)\end{bmatrix}^{\mathbf{T}}$ and $\boldsymbol{\epsilon}=\begin{bmatrix}\epsilon(1) & \ldots & \epsilon(N)\end{bmatrix}^T$. Then, (4) can be compactly written as

$$\mathbf{v}=\mathbf{H}(\boldsymbol{\omega_0})\boldsymbol{\gamma}+\boldsymbol{\epsilon} \tag{5}$$

where $\boldsymbol{\gamma}=\begin{bmatrix}A_1e^{jl_1\phi} & \ldots & A_Ke^{jl_K\phi}\end{bmatrix}^T$ and $\mathbf{H}(\omega_0)\in\mathbb{C}^{N\times K}$ denotes the Vandermonde matrix with the kth column given by $\mathbf{h_k}=\begin{bmatrix}e^{jl_k\omega_0} & e^{jl_k2\omega_0} & \ldots & e^{jl_kN\omega_0}\end{bmatrix}^T$. The problem of interest is to estimate ω_0, ϕ and A_k from the observations $\{v(n)\}_{n=1}^N$.

III. The New WLS Frequency and Phase Estimator

In this section, we extend the WLS [19] technique to our signal model, the rationale of the WLS is that it first ignores the harmonic structure and uses some standard sinusoidal parameter estimators, such as MUSIC [15] and ESPRIT [16], to obtain the initial estimates: $\tilde{\omega}_k$, $\tilde{\phi}_k$ and $\tilde{A}_k$ for ω_k, ϕ_k and A_k respectively, then the WLS utilizes the harmonic structure of the 3PH power system, i.e., $\omega_k=l_k\omega_0$ and $\phi_k=l_k\phi$, to refine the initial estimates.

Specifically, let $\boldsymbol{\zeta}=[l_1\phi,l_1\omega_0,\ldots,l_K\phi,l_K\omega_0]^T\in\mathbb{R}^{2K\times1}$ and $\boldsymbol{\eta}=[\phi,\omega_0]^T$. Then, there is a rank-two matrix $\mathbf{S}=\mathbf{l}\otimes\mathbf{I_2}$, where $\mathbf{l}=[l_1,\ldots,l_K]^T$, satisfying the equation

$$\boldsymbol{\zeta}=\mathbf{S}\boldsymbol{\eta}. \tag{6}$$

Let $\tilde{\boldsymbol{\zeta}}=[\tilde{\phi}_1,\tilde{\omega}_1,\ldots,\tilde{\phi}_K,\tilde{\omega}_K]^T$ be the corresponding initial estimate of $\boldsymbol{\zeta}$ from MUSIC. The WLS estimate $\hat{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}$ is given by

$$\hat{\boldsymbol{\eta}}=\arg\min_{\boldsymbol{\eta}}\|\tilde{\boldsymbol{\zeta}}-\mathbf{S}\boldsymbol{\eta}\|^{\mathbf{2}}_{\mathbf{W}}. \tag{7}$$

The weighting matrix $\mathbf{W}$ obtained from the CRB matrix without considering the harmonic structure is given by

$$\mathbf{W}=\begin{bmatrix} 2N\tilde{A}_k^2 & N^2\tilde{A}_k^2\\ N^2\tilde{A}_k^2 & \frac{2}{3}N^3\tilde{A}_k^2\end{bmatrix} \tag{8}$$

then, we rewrite the cost function in (7) as

$$\begin{aligned} J(\boldsymbol{\eta}) =&\|\tilde{\boldsymbol{\zeta}}-\mathbf{S}\boldsymbol{\eta}\|^{\mathbf{2}}_{\mathbf{W}}\\ =&\frac{3}{4\sigma^2}\sum_{k=1}^{K}\begin{bmatrix}\tilde{\phi}_k-l_k\phi\\ \tilde{\omega}_k-l_k\omega_0\end{bmatrix}^T\\ &\times\begin{bmatrix} 2N\tilde{A}_k^2 & N^2\tilde{A}_k^2\\ N^2\tilde{A}_k^2 & \frac{2}{3}N^3\tilde{A}_k^2\end{bmatrix}\begin{bmatrix}\tilde{\phi}_k-l_k\phi\\ \tilde{\omega}_k-l_k\omega_0\end{bmatrix}\\ =&\frac{3}{4\sigma^2}\sum_{k=1}^{K}\tilde{A}_k^2[2N(\tilde{\phi}_k-l_k\phi)^2\\ &+2N^2(\tilde{\phi}_k-l_k\phi)(\tilde{\omega}_k-l_k\omega_0)+\frac{2}{3}N^3(\tilde{\omega}_k-l_k\omega_0)^2]. \end{aligned} \tag{9}$$

The solution of (7) can be cast as follows: differentiating the above equation with respect to ϕ, and we have

$$\frac{dJ}{d\phi}=\frac{3}{4\sigma^2}\sum_{k=1}^{K}\tilde{A}_k^2\left[4Nl_k(l_k\phi-\tilde{\phi}_k)+2N^2l_k(l_k\omega_0-\tilde{\omega}_K)\right] \tag{10}$$

equating the expression to zero result in the estimate of ϕ as:

$$\hat{\phi}=\frac{\sum_{k=1}^{K}\tilde{A}_k^2l_k(2\tilde{\phi}_k+N\tilde{\omega}_k)}{\sum_{k=1}^{K}2\tilde{A}_k^2l_k^2}-\frac{N}{2}\omega_0 \tag{11}$$

where the right side of the equation depends on the unknown parameter ω_0. Substitute the above equation into (9) and take the differentiation w.r.t ω_0, yields the WLS estimate of ω_0:

$$\hat{\omega}_0=\frac{\sum_{k=1}^{K}l_k\tilde{A}_k^2\tilde{\omega}_k}{\sum_{k=1}^{K}l_k^2\tilde{A}_k^2} \tag{12}$$

using the above $\hat{\omega}_0$ to replace ω_0 in (11), we can obtain the WLS estimate of ϕ as:

$$\hat{\phi} = \frac{\sum_{k=1}^{K} l_k \tilde{A}_k^2 \tilde{\phi}_k}{\sum_{k=1}^{K} l_k^2 \tilde{A}_k^2}. \quad (13)$$

Hence, (12) and (13) provide closed-form expressions for the WLS estimates of the frequency and phase parameters. As $\hat{\omega}_0$ and $\hat{\phi}$ are weighted linear regression over $\{\tilde{\omega}_k\}_{k=1}^{K}$ and $\left\{\tilde{\phi}\right\}_{k=1}^{K}$, respectively, by utilizing the harmonic structure, WLS estimators can extract the useful information from the harmonic distortion and its performance is very close to the CRB (see Appendix A) with large observation data.

IV. Improved WLS for Power System with Limited Data

In this section, we first describe the disadvantage of applying the WLS estimator to a power system with limited observations, especially, only quarter cycle samples are available for the system. Then, an improved WLS (IWLS) technique based on iterative MUSIC algorithm is proposed.

A. Disadvantage of WLS in 3PH Power System

The WLS estimator performs well when the non-primary harmonics have comparable magnitudes with first order harmonic. Unfortunately, with very low harmonic distortion in 3PH power system, the WLS estimator loses its advantage in the data-limited case. The reason is that, with limited samples, the estimation accuracy for the unknown parameters of the non-primary harmonics with weak power is not guaranteed. Thus, these unreliable information used in the WLS estimator will jeopardize its estimation performance. In the following subsection, we present a scheme which can provide reliable initial estimation for the WLS estimator in the data-limited case.

B. Proposed IWLS Parameter Estimation

For the parameter estimation problem in a harmonic signal model, it is well known that the greater magnitude the harmonic has, the more accurate the associated parameter estimates are. An example to show the CRBs of the frequencies are given in Fig.1, where three harmonics are involved in the signal with independent frequencies $\omega_1, \omega_2, \omega_3$ and magnitudes $A_1 = 1, A_2 = 0.06$ and $A_3 = 0.05$, respectively. The observation data length is $N = 20$, and the CRB is computed in terms of $\frac{1}{\sigma^2}\mathbf{W}$, where σ^2 is the noise power and $\mathbf{W}$ is given in (8). It can be seen that a harmonic with the strongest magnitude has the lowest CRB for its frequency estimate. Therefore, the accuracy or the reliability of the three frequency estimates satisfies $acc(\hat{\omega}_1) > acc(\hat{\omega}_2) > acc(\hat{\omega}_3)$, where $acc(\hat{\omega})$ denotes the accuracy of $\hat{\omega}$. Based on this property, we propose an iterative algorithm to improve the estimation accuracy.

The proposed estimation scheme still involves a two-step procedure: initial estimation by using MUSIC and refined estimation by using WLS. Since the initial estimation for the unknown parameters of the harmonics with weak power

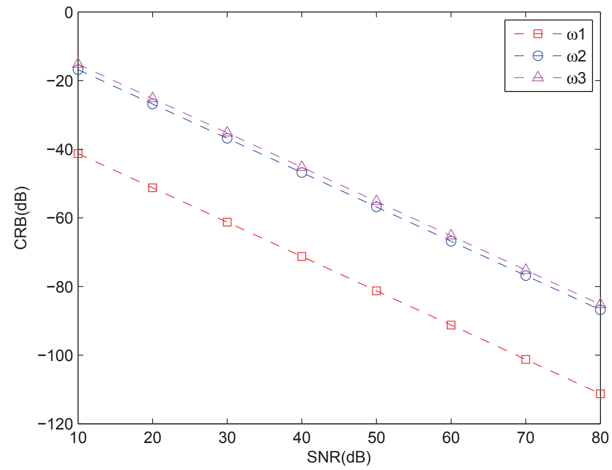


Fig. 1. CRBs of the frequencies versus SNR

is imprecise in the data-limited case, we introduce an iterative scheme to polish the initial estimates. Specifically, with the $N \times 1$ observation vector $\mathbf{v}$, we can use MUSIC to estimate the unknown parameters $\left\{\tilde{\omega}_k, \tilde{\phi}_k, \tilde{A}_k\right\}_{k=1}^{K}$ of all the K harmonics. However, we only save the most accurate and reliable estimate $\tilde{\boldsymbol{\tau}}_1 = \left[\tilde{\omega}_1, \tilde{\phi}_1, \tilde{A}_1\right]$ which corresponds to the harmonic with the strongest magnitude, i.e., the first order harmonic. Then, with $\tilde{\boldsymbol{\tau}}_1$ we can construct the first order harmonic component $\tilde{\mathbf{s}}(\tilde{\boldsymbol{\tau}}_1) = [\tilde{s}_1(1) \dots \tilde{s}_1(N)]^T$ with $\tilde{s}_1(n) = \tilde{A}_1 e^{j(n\tilde{\omega}_1 + \tilde{\phi}_1)}$ approximately. Applying MUSIC again on the vector $\mathbf{v}_1 = \mathbf{v} - \tilde{\mathbf{s}}(\tilde{\boldsymbol{\tau}}_1)$ which only involves $K-1$ harmonics, we can obtain the estimate vector $\tilde{\boldsymbol{\tau}}_2$ of the second strongest harmonic by saving the estimates corresponding to the strongest harmonic in the left $K-1$ harmonics. Continue the iteration with K steps, all the K polished initial estimates $\{\tilde{\boldsymbol{\tau}}_1, \dots, \tilde{\boldsymbol{\tau}}_K\}$ can be saved for the next refined estimation stage.

Remark: the MUSIC algorithm is often sensitive to the choice of M relative to the data length N, where M is the length of the data subvectors used in MUSIC. This is an inherent trade-off between having many subvectors in the averaging while retaining sufficient dimensions of the harmonics. We have observed that when $M = 4N/5$, MUSIC often provides the best estimation on the unknown parameters of the strongest harmonic. Therefore, we use $M = 4N/5$ for the MUSIC.

The structure of the proposed parameter estimation algorithm is shown in Fig.2, where $\mathbf{v} = [v(1) \dots v(N)]^T$ is the input observation vector and $\hat{\boldsymbol{\tau}}_0 = [\hat{\omega}_0, \hat{\phi}, \tilde{A}_1, \dots, \tilde{A}_K]$ is the output parameter estimates. The estimated kth harmonic component is given as $\tilde{\mathbf{s}}(\tilde{\boldsymbol{\tau}}_k) = [\tilde{s}_k(1) \dots \tilde{s}_k(N)]^T$ with $\tilde{s}_k(n) = \tilde{A}_k e^{j(n\tilde{\omega}_k + \tilde{\phi}_k)}$.

Recall that the WLS frequency estimator (12) and phase estimator (13) are weighted linear combination over $\{\tilde{\omega}_k\}_{k=1}^{K}$ and $\left\{\tilde{\phi}\right\}_{k=1}^{K}$, respectively. However, with limited observations or low SNR, not all of the K parameters can be estimated accurately. Thus, we only use the $k, k \leq K$ estimates of the

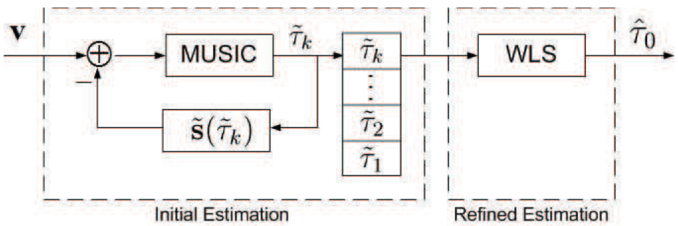


Fig. 2. Structure of the proposed IWLS parameter estimation algorithm

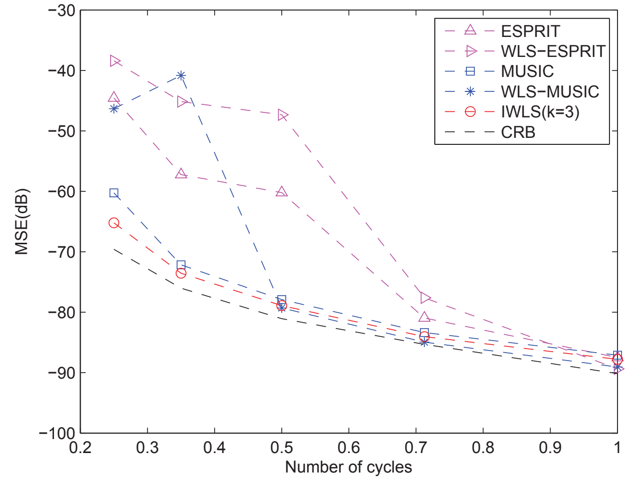
first k strongest harmonics to estimate ω_0 and ϕ. In other words, we only need to run k iteration steps rather than going through all K iterations. Note that we implement the iteration in the initial estimation only when the input observation data is limited. Therefore, the computational complexity introduced by the iterations is acceptable.

V. Numerical results

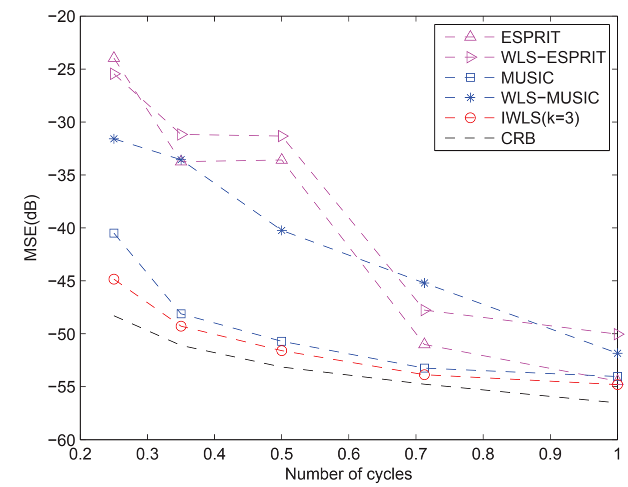
In this section, computer simulation results are presented to compare the performance of the proposed IWLS with other estimators, including MUSIC, ESPRIT, and their associated WLS estimators in the data-limited case, the CRB is also involved in the comparison.

The parameters of the 3PH power system (1) for performance test is set in terms of European Standard: EN 50160 [20]. The fundamental frequency and phase is $f_0 = 50$Hz and $\phi = 10°$ respectively. We consider the first six $(K = 6)$ strongest harmonics component in the signal and note that the multiples of 3rd harmonics are disappeared after Clarke transformation. Thus, the orders of the harmonics involved in the signal are $\{1, 5, 7, 11, 13, 17\}$ and the associated magnitudes are $\{1, 6\%, 5\%, 3.5\%, 3\%, 2\%\}$, the total harmonic distortion (THD) factor is $\text{THD} = 9.29\%$. The sample frequency used in the system is $f_s = 4$KHz, which indicates that there are totally 80 samples per cycle, and the fundamental angular frequency is $\omega_0 = 2\pi f_0/f_s = \pi/40$. The SNR for the system is defined as $10\log_{10} 3V_1^2/4\sigma^2$dB, where the coefficient $3/4$ is introduced by the Clarke transformtion.

We first compare the performance of the proposed IWLS with ESPRIT, MUSIC and their corresponding WLS. Fig.3(a) shows the mean square errors (MSE) of the fundamental frequency estimates along with the CRB as a function of the data length with SNR=40dB and $M = 4N/5$. It is seen that WLS frequency estimate can converge to the CRB with 0.5 cycle and 1 cycle data when using MUSIC and ESPRIT as initial estimation, respectively. However, WLS based on MUSIC and ESPRIT lose the advantage when the data length is less than half cycle. The proposed IWLS can achieve the best performance in small data case. Especially, when the data length is quarter cycle which is more interesting in practice, IWLS can have about 5dB improvement compared with MUSIC. Fig.3(b) presents the MSE of phase estimates, which also shows that the proposed IWLS performs best with quarter cycle data, while the WLS based on MUSIC and ESPRIT can even not converge to the CRB within one cycle.



(a)



(b)

Fig. 3. MSE of the parameter estimates and the associated CRB versus the data length with SNR=40dB, $M = 4N/5$. (a) Estimates of fundamental frequency ω_0. (b) Estimates of phase ϕ.

We also evaluate the estimation performance of the IWLS in terms of the phase angle. The iteration steps in IWLS is $k = 3$, and the data length for each estimate is $N = 20$ (quarter cycle). Fig.4 provides the phase angle estimates with a single run for the first 50 ms. We can see that the proposed method perfectly tracks the actual phase angle after quarter cycle. The corresponding MSE is also shown in this figure where 100 trials are tested and the simulation time period is 500ms for each run. It shows that the MSE is below -35dB in most of the time, which means that the proposed algorithm can estimate the phase angle with high accuracy.

VI. Conclusion

We have considered a fundamental frequency and phase estimation problem in 3PH power system with quarter cycle samples. The model of 3PH power system with harmonic distortion is first convert to a noise-corrupted single-phase harmonic signal model. Then, we compute a new WLS frequency estimator and phase estimator. Due to the low voltage

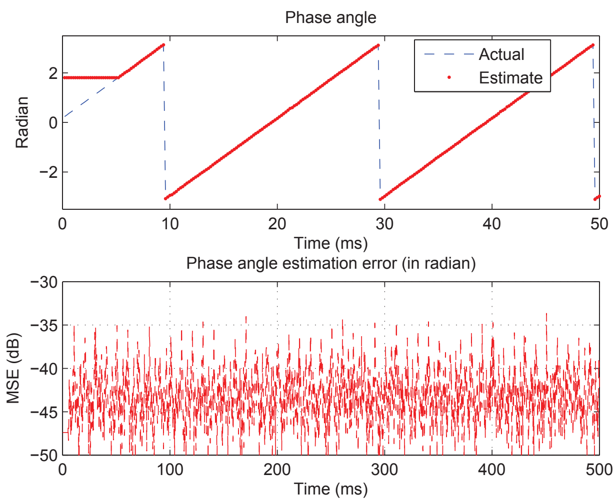


Fig. 4. Phase angle estimation and its associate MSE based on quarter cycle data

characteristic of the non-primary harmonics and the limited observations constraint in realistic 3PH power system, an improved WLS which using an iterative algorithm to polish the initial estimates for WLS is introduced. The statistical performance of the new method has been evaluated and compared to the conventional estimation in terms of the mean square error. The simulations show that the performance gain by applying the proposed IWLS estimator is significant when only quarter cycle samples are available.

Appendix A
Cramer-Rao Bound

In this appendix, the CRB for the parameter estimation problem posed for the complex data model in (4) is derived. By defining $\bar{\boldsymbol{\eta}} = [\phi, \omega_0, \mathbf{a}^T]^T \in \mathbb{R}^{(K+2)\times 1}$ and $\mathbf{a}^T = [A_1, \ldots, A_K]$ we rewrite (5) as

$$\mathbf{v} = \mathbf{B}(\omega_0, \phi)\mathbf{a} + \boldsymbol{\epsilon} = \mathbf{x}(\bar{\boldsymbol{\eta}}) + \boldsymbol{\epsilon} \quad (14)$$

where $\mathbf{B}(\omega_0, \phi) \in \mathbb{C}^{N\times K}$ with its kth column is defined by $\mathbf{b}_k = e^{jl_k\phi}\mathbf{h}_k$. By using the Slepian-Bangs formula, the CRB matrix for the problem under study is given by

$$\mathbf{CRB}^{-1}(\bar{\boldsymbol{\eta}}) = \frac{3}{2\sigma^2}\Re\left[\frac{\partial \mathbf{x}^H(\bar{\boldsymbol{\eta}})}{\partial \bar{\boldsymbol{\eta}}}\frac{\partial \mathbf{x}(\bar{\boldsymbol{\eta}})}{\partial \bar{\boldsymbol{\eta}}^T}\right]. \quad (15)$$

We then evaluate the partial derivatives as follows:

$$\frac{\partial \mathbf{x}(\bar{\boldsymbol{\eta}})}{\partial \bar{\boldsymbol{\eta}}^T} = \left[\frac{\partial \mathbf{x}(\bar{\boldsymbol{\eta}})}{\partial \phi}, \frac{\partial \mathbf{x}(\bar{\boldsymbol{\eta}})}{\partial \omega_0}, \frac{\partial \mathbf{x}(\bar{\boldsymbol{\eta}})}{\partial \mathbf{a}^T}\right] \quad (16)$$

where

$$\frac{\partial \mathbf{x}(\bar{\boldsymbol{\eta}})}{\partial \phi} = \mathbf{Ca} \quad (17)$$

$$\frac{\partial \mathbf{x}(\bar{\boldsymbol{\eta}})}{\partial \omega_0} = \mathbf{Da} \quad (18)$$

$$\frac{\partial \mathbf{x}(\bar{\boldsymbol{\eta}})}{\partial \mathbf{a}^T} = \mathbf{B} \quad (19)$$

with the kth columns of $\mathbf{C} \in \mathbb{C}^{N\times K}$ and $\mathbf{D} \in \mathbb{C}^{N\times K}$ given by $\mathbf{c}_k = jl_k\mathbf{b}_k$ and $\mathbf{d}_k = jl_k e^{jl_k\phi}[e^{jl_k\omega_0}, 2e^{jl_k 2\omega_0}, \ldots, Ne^{jl_k N\omega_0}]^T$, respectively. Finally, it follows that

$$\mathbf{CRB}(\bar{\boldsymbol{\eta}}) = \frac{2\sigma^2}{3}\left\{\Re\begin{bmatrix}\mathbf{a}^H\mathbf{C}^H\mathbf{Ca} & \mathbf{a}^H\mathbf{C}^H\mathbf{Da} & \mathbf{a}^H\mathbf{C}^H\mathbf{B}\\ \mathbf{a}^H\mathbf{D}^H\mathbf{Ca} & \mathbf{a}^H\mathbf{D}^H\mathbf{Da} & \mathbf{a}^H\mathbf{D}^H\mathbf{B}\\ \mathbf{B}^H\mathbf{Ca} & \mathbf{B}^H\mathbf{Da} & \mathbf{B}^H\mathbf{B}\end{bmatrix}\right\}^{-1} \quad (20)$$

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