MITSUBISHI ELECTRIC RESEARCH LABORATORIES
http://www.merl.com

On Embedding the Angles Between Signals

Boufounos, P.T.

TR2013-061 July 2013

Abstract

The phase of randomized complex-valued projections of real signals preserves information about the angle, i.e., the correlation, between signals. This information can be exploited to design angle preserving embeddings, which represent such correlations. These embeddings generalize known results on binary embeddings and 1-bit compressive sensing and reduce the embedding uncertainty.

2013 Signal Processing with Adaptive Sparse Structural Representation (SPARS)

On Embedding The Angles Between Signals

Petros T Boufounos
Mitsubishi Electric Research Laboratories, Cambirdge, MA 02139, USA, *petrosb@merl.com*

***Abstract*—The phase of randomized complex-valued projections of real signals preserves information about the angle, i.e., the correlation, between signals. This information can be exploited to design angle-preserving embeddings, which represent such correlations. These embeddings generalize known results on binary embeddings and 1-bit compressive sensing and reduce the embedding uncertainty.**

Randomized embeddings play an increasingly important role in signal processing applications. Such embeddings transform a signal space to another of typically lower dimension or convenient computational properties. The embedding approximately preserves some aspect of the signal geometry, such that operations on the embedded signals directly map to operations on the original signals.

The most celebrated embeddings are Johnson-Lindestrauss (J-L) embeddings, preserving ℓ_2 distances [1]. They are functions $f : \mathcal{S} \rightarrow \mathbb{R}^K$ mapping a finite set of L signals $\mathcal{S} \subset \mathbb{R}^N$ to a K-dimensional space such that, given two signals $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{S}$, their images satisfy:

$$(1-\epsilon)\|\mathbf{x}-\mathbf{y}\|_2^2 \leq \|f(\mathbf{x})-f(\mathbf{y})\|_2^2 \leq (1+\epsilon)\|\mathbf{x}-\mathbf{y}\|_2^2.$$

In many applications, however, ℓ_2 distance is not the appropriate distance metric. In this work we explore embeddings of signals that preserve angles, i.e., correlations. The remainder of this development uses the normalized angle defined between two signals $\mathbf{x}$ and $\mathbf{x}$' as

$$d_\angle = \frac{1}{\pi} \arccos \frac{\langle \mathbf{x}, \mathbf{x}' \rangle}{\|\mathbf{x}\| \|\mathbf{x}'\|} \quad (1)$$

Often, angles between signals are more informative than distances. Furthermore, if signals are normalized, their angle captures their distance. Embeddings preserving angles instead of distances can be more efficient in such cases.

Angle embeddings were first introduced in the context of 1-bit compressive sensing [2]. The binary ϵ-stable embedding encodes signals using a random projection followed by a 1-bit scalar quantizer that only preserves the sign of each projection coefficient:

$$\mathbf{q} = \mathrm{sign}(\mathbf{A}\mathbf{x}). \quad (2)$$

The normalized angle between two signals $\mathbf{x}$ and $\mathbf{x}'$ embedded in $\mathbf{q}$ and $\mathbf{q}'$, respectively, is preserved in the normalized hamming distance between their embeddings, as follows:

$$|d_H(\mathbf{q}, \mathbf{q}') - d_\angle(\mathbf{x}, \mathbf{x}')| \leq \epsilon, \quad (3)$$

where $d_H(\mathbf{x}, \mathbf{x}') = (\sum_i x_i \oplus x_i')/K$ denotes the normalized hamming distance between the signal embeddings.

Instead, in this work we consider continuous embeddings, obtained by first projecting the signal to a complex-valued space and only preserving the phase of the projection coefficients:

$$\mathbf{y} = \angle(\mathbf{A}_{\mathbb{C}}\mathbf{x}), \quad (4)$$

where $\mathbf{A}_{\mathbb{C}} \in \mathbb{C}^{K \times N}$ is a randomly drawn matrix with i.i.d. elements drawn from the standard complex normal distribution.

To demonstrate that angles are preserved we should show that given a pair of signals $\mathbf{x}$, $\mathbf{x}'$, the expected value of the phase difference of their complex projection coefficients is proportional to their angle

$$E\left\{\left|\frac{1}{\pi}\angle\left(e^{i(y_i - y_i')}\right)\right|\right\} = d_\angle(\mathbf{x}, \mathbf{x}'), \quad (5)$$

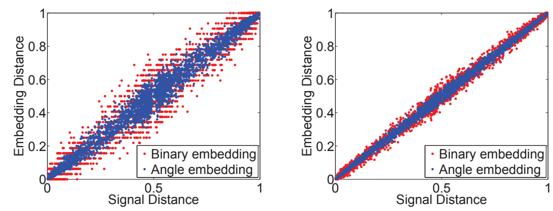

Fig. 1. Angle embeddings (blue dots) capture the true angle between signals with less uncertainty compared to binary embeddings (red dots). Plots generated using for $K = 32$ (left) or $K = 256$ (right) and $N = 1024$.

where $\angle(\cdot)$ measures the principal phase of a complex number (used here to appropriately take wrapping into account). A concentration of measure argument using Hoeffding's inequality can then show that

$$\left|\frac{1}{K}\sum_i \left|\frac{1}{\pi}\angle\left(e^{i(y_i - y_i')}\right)\right| - d_\angle(\mathbf{x}, vx')\right| \leq \epsilon \quad (6)$$

with probability greater than $1 - 2e^{2\log L - 2\epsilon^2 K}$. In other words, similar to J-L embeddings, $K = O(\log L)$ dimensions are sufficient to embed a cloud of L points. The argument can be extended to infinite sets such as sparse signals, using methods similar to [3].

The proposed embeddings can be considered a generalization of 1-bit embeddings as the phase of complex signals generalizes the sign of real-valued ones. Similar to 1-bit embeddings, phase embeddings eliminate magnitude information but preserve the remaining information about the signal which allows angle computation. Figure 1 compares the two embeddings, plotting simulation results on pairs of signals with different angles, as embedded in a space of lower (left) and higher (right) dimensional space using binary (red) or continuous angle (blue) embeddings. Angle embeddings capture the true angles with much less uncertainty compared to binary embeddings for the same measurements K. If the bit-rate is important, the phase can be quantized and methods similar to [4] can be used for rate allocation.

Phase-only compressive sensing, i.e., sparse reconstruction, is also possible, using a program similar to [5]. Details can be found in [6].

References

[1] W. Johnson and J. Lindenstrauss, "Extensions of Lipschitz mappings into a Hilbert space," *Contemporary Mathematics*, vol. 26, pp. 189 –206, 1984.

[2] L. Jacques, J. N. Laska, P. T. Boufounos, and R. G. Baraniuk, "Robust 1-bit compressive sensing via binary stable embeddings of sparse vectors," *Trans. Info. Thery*, 2013, to appear.

[3] R. Baraniuk, M. Davenport, R. DeVore, and M. Wakin, "A simple proof of the restricted isometry property for random matrices," *Const. Approx.*, vol. 28, no. 3, pp. 253–263, 2008.

[4] M. Li, S. Rane, and P. T. Boufounos, "Quantized embeddings of scale-invariant image features for mobile augmented reality," in *Proc. IEEE International Workshop on Multimedia Signal Processing (MMSP)*, Banff, Canada, Sept. 17–19 2012.

[5] Y. Plan and R. Vershynin, "One-bit compressed sensing by linear programming," *Comm. Pure and Applied Math.*, 2012, to appear.

[6] P. T. Boufounos, "Sparse signal reconstruction from phase-only measurements," in *Proc. 10th International Conference on Sampling Theory and Applications (SampTA)*, Bremen, Germany, July 1-5 2013.