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TR2013-107 December 2013

Abstract

We study the problem of tracking a time-varying reference signal for constrained linear systems. The reference signal is the output of a linear system driven by an unknown bounded input. The goal is to track the reference signal and never violate a predefined tracking error bound. The paper presents the design of a reference tracking controller satisfying state and input constraints and guaranteeing the desired tracking error bound for all admissible reference signals. A model predictive controller (MPC) enforcing a robust invariant set is employed. We show how to compute the robust invariant set and how to design the tracking MPC law which guarantees constraints satisfaction and persistent feasibility. Simulations show the effectiveness of the proposed approach.

IEEE Conference on Decision and Control (CDC)

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Stefano Di Cairano, Francesco Borrelli

***Abstract*— We study the problem of tracking a time-varying reference signal for constrained linear systems. The reference signal is the output of a linear system driven by an unknown bounded input. The goal is to track the reference signal and never violate a predefined tracking error bound. The paper presents the design of a reference tracking controller satisfying state and input constraints and guaranteeing the desired tracking error bound for all admissible reference signals. A model predictive controller (MPC) enforcing a robust invariant set is employed. We show how to compute the robust invariant set and how to design the tracking MPC law which guarantees constraints satisfaction and persistent feasibility. Simulations show the effectiveness of the proposed approach.**

I. INTRODUCTION

The design of reference tracking controllers enforcing inputs and state constraints has mainly followed two approaches. The approaches proposed in [1]–[8] are based on model predictive control (MPC). MPC [9] uses a model of the plant to predict the future evolution of the system and to compute the optimal control strategy along a finite future horizon. At each time step MPC optimizes a performance index over a sequence of future input commands subject to the system operating constraints. The first element of the optimal sequence is the control action applied to the plant. At the next time step a new optimization problem over a shifted prediction horizon is solved. Research on reference tracking MPC (see e.g., [1]–[8]) has focused on guaranteeing asymptotic offset-free control for references that belong to a specific class of signals when the constraints are inactive. These approaches differ in the class of reference signals, in the assumptions that guarantee offset-free control, and in the MPC problem setup. In the aforementioned literature the reference signal is usually the output of an *autonomous* linear system.

An alternative approach to design reference tracking controllers for constrained systems is based on reference governor [10]–[13]. A reference governor is a control algorithm that modifies the reference signal as a function of the system state to generate a virtual reference. The virtual reference is then tracked by an existing linear controller. If the reference governor is properly designed, the closed-loop system satisfies state and input constraints, is asymptotically stable, and exhibits asymptotic offset-free tracking [11]–[13].

In this paper we aim at designing a reference tracking controller that satisfies state and input constraints and guarantees a predefined bound on tracking error during both *transient and steady-state*. The tracking error bound is guaranteed at all times for a class of reference signals which is more general than what can be represented by the output of a linear *autonomous* system. In particular, the reference signal is the output of the "reference generator". The reference generator is a constrained linear system *driven by an unknown input* belonging to a known bounded set. A bounded reference signal with a bounded first derivative is the simplest class of reference signal that can be modeled with this approach [14, p. 159]. The recent work in [15] applies a similar reference generator approach and characterizes the set of states to which the tracking error converges. As opposed to reference governor and virtual setpoint-augmented MPC [7], [15], the proposed controller is not allowed to modify the reference during execution.

In the proposed approach we compute the set of (plant and reference) states for which there exists a control law guaranteeing constraint satisfaction and the desired bound on tracking error at all times and for all admissible references. Such *robust control invariant set* is then enforced in a model predictive control strategy. If the MPC problem is initially feasible, the control strategy ensures that the system constraints and the desired tracking error bound are satisfied at all future steps, and for all admissible reference trajectories produced by the reference generator. This paper shows how to efficiently compute the robust control invariant set and how to design the tracking MPC controller in order to guarantee feasibility at all time instants.

The proposed control design is relevant in a number of practical applications. For instance, in dual-stage automated tooling machines, a slow stage with large workspace (also called operating range) moves a stage that actuates the tool with much faster dynamics and a limited workspace. Thus, the tool workspace is the sum of the stages' workspace. Because of the stages timescale separation (often 2-3 orders of magnitude), to ensure tool trajectory tracking it is enough to control the slow stage so that the tool position reference is always within the workspace of the fast stage. The methodology presented in this paper can be used to control the slow stage so that it never violates the error bound consisting of the fast stage workspace, for all admissible tool reference trajectories. The automotive and aerospace industry are also rich of potential applications. For instance, the desired engine torque in SI engines is obtained by controlling airflow and spark timing [16]. Due to limited spark timing authority, the the airflow-generated torque must be controlled in a range around the requested torque. In hybrid electric vehicles the combustion engine power needs to be controlled so that the difference with the driver-requested power can be achieved

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by electric power [17]. Similar applications can be found also in vehicle stability control with multiple actuators [18], [19].

After reviewing some basic results in control invariant sets and formalizing the problem in Section II, in Section III we discuss how to synthesize the robust control invariant set for tracking. In Section IV we discuss the design of the tracking MPC controller that guarantees persistent feasibility, constraints satisfaction, and the desired tracking error bounds. Finally, in Section V we show an example validating the proposed approach, and in Section VI we draw our conclusions and discuss future work.

Notation: $\mathbb{R}$, $\mathbb{R}_{0+}$, $\mathbb{R}_{+}$ are the sets of real, nonnegative real, positive real numbers, and $\mathbb{Z}$, $\mathbb{Z}_{0+}$, $\mathbb{Z}_{+}$ are the sets of integer, nonnegative integer, positive integer numbers. Unless otherwise specified, $\|\cdot\|$ indicates either the 1 or ∞-norm, and $\mathcal{B}(a,\rho)$ where $a \in \mathbb{R}^n$, $\rho \in \mathbb{R}_+$ consistently indicates either the 1 or ∞-norm ball in $\mathbb{R}^n$, centered in a and of radius ρ. The results of this paper hold for consistently using either 1-norm, or ∞-norm. For a discrete time signal $x \in \mathbb{R}^n$ with sampling period T_s, x_t is the state a sampling instant t, i.e., at time $T_s t$. The notation $x_{k|t}$ denotes the predicted value of x at sample $t+k$, i.e., x_{t+k}, based on data at sample t, and by definition, $x_{0|t} = x_t$. By $[x]_i$ we denote the i-th component of x, and, by I and 0 the identity and the "all-zero" matrices of appropriate size.

II. Preliminaries and Problem Definition

In this section, first we review some of useful results on robust control invariant sets, then we formalize the problem tackled in this paper. The following basic definitions and results in robust control invariant sets for constrained systems can be found in [20]–[23]. See [24] for a comprehensive survey on set invariance in control.

A. Preliminaries on Invariant Sets

Consider the system

$$x_{t+1} = f(x_t, u_t, w_t), \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^d$ are the state, input and disturbance vectors, respectively, subject to the constraints

$$x_t \in \mathcal{X},\ u_t \in \mathcal{U},\ w_t \in \mathcal{W},\ \forall t \in \mathbb{Z}_{0+}. \tag{2}$$

A robust control invariant set is a set of states for which there exists a control law such that (1) never violates (2) for any admissible sequence of disturbances.

Definition 1 (Robust control invariant set): A set $\mathcal{C} \subseteq \mathcal{X}$ is said to be a robust control invariant set for (1) if

$$\begin{aligned} & x_t \in \mathcal{C} \Rightarrow \exists u_t \in \mathcal{U} : \\ & \qquad f(x_t, u_t, w_t) \subseteq \mathcal{C},\ \forall w_t \in \mathcal{W},\ \forall t \in \mathbb{Z}_{0+}. \end{aligned} \tag{3}$$

The set $\mathcal{C}^\infty$ is said to be the *maximal* robust control invariant if it is a robust control invariant set and contains all the other robust control invariant sets in $\mathcal{X}$. □

The computation of robust control invariant sets relies on the **Pre**-set operator

$$\begin{aligned} \mathbf{Pre}(\mathcal{S}, \mathcal{W}) \ \triangleq\ & \{x \in \mathcal{X} \ :\ \exists u \in \mathcal{U} \\ & f(x, u, w) \subseteq \mathcal{S},\ \forall w \in \mathcal{W}\}, \end{aligned} \tag{4}$$

which computes the set of states of (1) that can be robustly driven to the target set $\mathcal{S} \in \mathbb{R}^n$ in one step.

The procedure to compute the maximal robust control invariant set for system (1) subject to constraints (2) based on the operator (4) is summarized by the following algorithm.

Algorithm 1 (Computation of $\mathcal{C}^\infty$):

1) $\Omega_0 \leftarrow \mathcal{X}$
2) $\Omega_{k+1} \leftarrow \mathbf{Pre}(\Omega_k, \mathcal{W})$
3) `If` $\Omega_{k+1} = \Omega_k$
   $\mathcal{C}^\infty \leftarrow \Omega_{k+1}$, `return`
4) $k \leftarrow k+1$, `goto` 2.

□

Algorithm 1 generates the sequence of sets $\{\Omega_k\}_{k=0}^{n_\omega}$, $n_\omega \in \mathbb{Z}_{0+}$, satisfying $\Omega_{k+1} \subseteq \Omega_k$, for all $k \in \mathbb{Z}_{0+}$. Algorithm 1 terminates if $\Omega_{k+1} = \Omega_k$, and in this case Ω_k is the maximal robust control invariant set $\mathcal{C}^\infty$ for (1) subject to (2). See [20], [21] for details on termination of Algorithm 1.

Definition 2 (Input admissible set for $\mathcal{C}$): Given a robust control invariant set $\mathcal{C}$ for (1)-(2), the input admissible set for state $x \in \mathcal{C}$ is

$$\mathcal{C}_u(x) = \{u \in \mathcal{U} : f(x, u, w) \in \mathcal{C},\ \forall w \in \mathcal{W}\}.$$

□

Definition 3: If system (1) is not subject to exogenous disturbances (i.e., $\mathcal{W} = \emptyset$), the set $\mathcal{C}$ in Definition 1 is called simply "control invariant set".

□

Next, we formally define the problem tackled in this paper.

B. Problem definition

Consider the system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \\ y_t &= Cx_t, \end{aligned} \tag{5}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output vectors, respectively. System (5) is subject to the constraints

$$x_t \in \mathcal{X},\ u_t \in \mathcal{U},\ \forall t \in \mathbb{Z}_{0+}. \tag{6}$$

We study the problem of tracking with a predefined error bound the time-varying reference signal y_t^r generated by the reference model

$$\begin{aligned} r_{t+1} &= A_r r_t + B_r \gamma_t \\ y_t^r &= C_r r_t, \end{aligned} \tag{7}$$

where $r \in \mathbb{R}^{n_r}$, $\gamma \in \mathbb{R}^{m_r}$ and $y^r \in \mathbb{R}^p$ are the state, input, and output vectors of the reference model, respectively.

The reference model (7) is subject to the constraints

$$r_t \in \mathcal{R},\ \gamma_t \in \Gamma, \tag{8}$$

and in what follows we assume that γ, the input to the reference model (7), is selected by a *reference generator algorithm* so that constraints (8) are satisfied.

At every time instant $t \in \mathbb{Z}_{0+}$, the reference generator algorithm uses the current feasible reference r_t to compute a $\gamma_t \in \Gamma$ such that the reference r_{t+1} at time $t+1$ is admissible, i.e., $r_{t+1} \in \mathcal{R}$. Thus, r_t must belong to a control invariant set $\mathcal{C}_r$ for (7), (8), and γ_t needs to be chosen accordingly. By the definitions in Section II-A, the control invariant set $\mathcal{C}_r$ for (7)-(8) is such that

$$r_t \in \mathcal{C}_r \Rightarrow \; \exists \gamma_t \in \Gamma : \; r_{t+1} \in \mathcal{C}_r, \; \forall t \in \mathbb{Z}_{0+}. \tag{9}$$

We denote by $\mathcal{C}_\gamma(r)$ the input admissible set associated to $\mathcal{C}_r$ in (9).

Starting from $r_0 \in \mathcal{C}_r$, the reference generator algorithm produces feasible references $r_t \in \mathcal{C}_r$, for all $t \in \mathbb{Z}_{0+}$, by choosing $\gamma_t \in \mathcal{C}_\gamma(r_t)$ as summarized by the following assumption.

Assumption 1: At every $t \in \mathbb{Z}_{0+}$, given $r_t \in \mathcal{C}_r$, where $\mathcal{C}_r$ is a known control invariant set of (7)-(8), the reference generator algorithm enforces $r_{t+1} \in \mathcal{C}_r$ by selecting $\gamma_t \in \mathcal{C}_\gamma(r_t)$. □

Note that if the reference system (7) is an integrator ($A_r = 1$, $B_r = 1$, $C_r = 1$) then

$$\mathcal{C}_r^\infty = \mathcal{R}, \quad \mathcal{C}_\gamma(r) = \{\gamma \in \Gamma \; | r + \gamma \in \mathcal{R}\}.$$

Remark 1: The bounds on r induce state-dependent bounds on γ because, for instance, when far from the border of $\mathcal{R}$ (or $\mathcal{C}_r$) not all the values of Γ are admissible for γ. Indeed, r needs to be bounded otherwise the bounded system (5) will not be able to always track it with a bounded error. Furthermore, r, x needs in general to be bounded for the computation of the robust control invariant set to converge.

The results of this paper do not depend on the chosen reference generator algorithm, as long as (8) is satisfied. However, it is important to notice that the reference generator algorithm is separated from the tracking controller. As opposed to [7], [15], the reference cannot be modified by the controller. In addition, the reference generator algorithm is unaware of the system state and constraints, as opposed to the reference governor approaches [10]–[12], and its only purpose is to guarantee that the reference signal is satisfying its own constraints (8) at any time instant.

The problem we address in this paper is formalized as follows.

Problem 1: Consider system (5) subject to constraints (6), reference model (7) subject to constraints (8), and the tracking error bound $\epsilon \in \mathbb{R}_+$. Let $R_t^N = [r'_{0|t}, \ldots, r'_{N|t}]'$, $N \in \mathbb{Z}_{0+}$ be a reference profile satisfying Assumption 1. Design a control law $u_t = \kappa(x_t, R_t^N)$, such that system (5) in closed-loop with $\kappa(x_t, R_t^N)$ satisfies constraints (6) and guarantees

$$\|y_t - y_t^r\| \leq \epsilon, \tag{10}$$

for all admissible references $r_t \in \mathcal{R}$ and for every $t \in \mathbb{Z}_{0+}$. We denote by $\mathcal{X}_0$ the set of initial states and references $[x_0', r_0']'$ for which $\kappa(x_t, R_t^N)$ solves Problem 1. □

In Problem 1 the controller may have access to a predicted future reference profile of length $N \in \mathbb{Z}_{0+}$. However, there is no guarantee that such reference preview is correct, that is, it is possible that $r_{t+k} \neq r_{k|t}$. In this way our results can be used also for applications where the reference preview may change in real-time, possibly unexpectedly.

III. Robust Control Invariant Set for Bounded Error Tracking

Consider (5) subject to (6), (7) subject to (8), and define

$$\mathcal{X}_{x,r} = \{[x' \; r']' \in \mathbb{R}^n \times \mathbb{R}^{n_r} : x \in \mathcal{X}, \; r \in \mathcal{R}, \\ (Cx - C_r r) \in \mathcal{B}(0, \epsilon)\}. \tag{11}$$

At any time $t \in \mathbb{Z}_{0+}$, given $[x_t' \; r_t']' \in \mathcal{X}_{x,r}$, the control law that solves Problem 1 must guarantee that $[x_{t+1}' \; r_{t+1}']' \in \mathcal{X}_{x,r}$ for every admissible reference. To this end we construct $\mathcal{C}_{x,r} \subseteq \mathcal{X}_{x,r}$ that is a robust control invariant set for (5)-(7) subject to (6), (8), (10), and robust to any $\gamma \in \mathcal{C}_\gamma(r)$. $\mathcal{C}_{x,r}$ is such that

$$\mathcal{C}_{x,r} \subseteq \mathcal{X}_{x,r}, \tag{12a}$$

$$[x' \; r']' \in \mathcal{C}_{x,r} \Rightarrow \exists u \in \mathcal{U} : \tag{12b}$$

$$[Ax + Bu, A_r r + B_r \gamma] \in \mathcal{C}_{x,r}, \; \forall \gamma \in \mathcal{C}_\gamma(r).$$

The implementation of Algorithm 1 for systems (5), (7) subject to (6), (8) and (10) requires the computation of

$$\textbf{Pre}(\Omega_k, \mathcal{C}_\gamma(r)) = \{[x' \; r']' \in \mathcal{X}_{x,r} : \; \exists u \in \mathcal{U}, \\ [(Ax + Bu)', (A_r r + B_r \gamma)']' \in \Omega_k, \forall \gamma \in \mathcal{C}_\gamma(r)\}, \tag{13}$$

where $\Omega_k \subseteq \mathcal{X}_{x,r}$. The computation of (13) is involved even if Ω_k, $\mathcal{C}_\gamma(r)$, $\mathcal{X}_{x,r}$ and $\mathcal{U}$ are polyhedral. In fact γ, which has the effect of a disturbance, belongs to the state dependent set $\mathcal{C}_\gamma(r)$. In [25], [26], an algorithm was proposed to compute (13), that results in an invariant set described as union of polyhedra, which is, in general, non-convex. We propose next an algorithm which exploits the decoupling of the dynamics (5) and (7) to compute a *polyhedral* invariant set $\mathcal{C}_{x,r}$. The resulting $\mathcal{C}_{x,r}$ is, in general, a subset of the maximal one obtained from the algorithms in [25], [26].

Algorithm 2 (Computation of $\mathcal{C}_{x,r}$):

1) `Initialization:` $k = 0$, $M_0 = C$, $L_0 = C_r$, $\bar{\mathcal{B}}_0 = \mathcal{B}(0, \epsilon)$, $\mathcal{X}_0 = \mathcal{X}$,

$$\bar{\Omega}_0 \;\; = \;\; \{[x' \; r']' \in \mathcal{X}_{x,r} : \; x \in \mathcal{X}, r \in \mathcal{C}_r, \\ (Cx - C_r r) \in \mathcal{B}(0, \epsilon)\}. \tag{14}$$

2) write $\bar{\Omega}_k$ as

$$\bar{\Omega}_k \;\; = \;\; \{[x' \; r']' \in \mathcal{X}_{x,r} : \; x \in \mathcal{X}_k, \; r \in \mathcal{C}_r, \\ (M_k x - L_k r) \in \bar{\mathcal{B}}_k\}. \tag{15}$$

3) $\bar{\Omega}_{k+1} = \textbf{Pre}_f(\bar{\Omega}_k, \Gamma)$ where

$$\begin{aligned} \textbf{Pre}_f(\bar{\Omega}_k, \Gamma) \;\; = \; & \\ \{[x' \; r']' \in \mathcal{X}_{x,r} : \; & \exists u \in \mathcal{U}, Ax + Bu \in \mathcal{X}_k, \\ r \in \mathcal{C}_r, & \\ M_k(Ax + Bu) - & L_k(A_r r + B_r \gamma) \in \bar{\mathcal{B}}_k, \\ \forall \gamma \in \Gamma\}. & \end{aligned} \tag{16}$$

4) `if` $\bar{\Omega}_{k+1} = \bar{\Omega}_k$
   `then` $\mathcal{C}_{x,r} \leftarrow \bar{\Omega}_{k+1}$, `return`
5) $M_{k+1} = M_k A$, $L_{k+1} = L_k A_r$,

$$\begin{aligned}\mathcal{X}_{k+1} &= \textbf{Pre}(\mathcal{X}_k)\\ &= \{x \in \mathcal{X} :\ \exists u \in \mathcal{U}, Ax + Bu \in \mathcal{X}_k\},\end{aligned}$$

$$\begin{aligned}\bar{\mathcal{B}}_{k+1} = \{v \in \mathbb{R}^p :\ &\exists u \in \mathcal{U},\ \exists b \in \bar{\mathcal{B}}_k\\ &v = b - M_k Bu + L_k B_r \gamma,\\ &\forall \gamma \in \Gamma\}\end{aligned} \tag{17}$$

6) $k = k + 1$, `goto 2`.

□

Theorem 1: Let Assumption 1 hold, and let Algorithm 2 converge in a finite number of iterations. Then, the output $\mathcal{C}_{x,r}$ of Algorithm 2 is a robust control invariant set for (5), (7) subject to (6), (8), (10) and robust to any $\gamma_r \in \mathcal{C}_\gamma(r)$.

Proof (sketch): Due to limited space we present the logical structure of the proof. Details will be reported in full in an extended publication. Let $\mathcal{X}_0 = \mathcal{X}$, $\mathcal{R}_0 = \mathcal{R}$, and $\mathcal{R}_{k+1} = \textbf{Pre}(\mathcal{R}_k, \mathcal{C}_\gamma(r)) = \{r \in \mathcal{R} :\ A_r r + B_r \gamma \in \mathcal{R}_k,\ \forall \gamma \in \mathcal{C}_\gamma(r)\}$.

Consider Algorithm 1 (based on (13)) and Algorithm 2, and the corresponding sequences of sets $\{\Omega_0,\ \Omega_1,\ \ldots\}$ and $\{\bar{\Omega}_0,\ \bar{\Omega}_1,\ \ldots\}$, respectively. The proof uses induction arguments. Assume that at step k:

1) Ω_k can be written as

$$\begin{aligned}\Omega_k = \{[x'\ r']' \in \mathcal{X}_{x,r} : x \in \mathcal{X}_k,\ r \in \mathcal{R},\\ (M_k x - L_k r) \in \mathcal{B}_k\},\end{aligned} \tag{18}$$

where

$$\begin{aligned}\mathcal{B}_{k+1} = \{v \in \mathbb{R}^p :\ &\exists u \in \mathcal{U},\ \exists b \in \mathcal{B}_k\\ &v = b - M_k Bu + L_k B_r \gamma,\ \forall \gamma \in \mathcal{C}_\gamma(r)\}\end{aligned}$$

2) $\bar{\Omega}_k$ can be written as (15)
3) $\bar{\Omega}_k \subseteq \Omega_k$.

We prove that: (*i*) Ω_{k+1} can be written as in (18), (*ii*) $\bar{\Omega}_{k+1}$ can be written as in (15), and (*iii*) $\bar{\Omega}_{k+1} \subseteq \Omega_{k+1}$. ■

Remark 2: The modifications to Algorithm 1 to obtain Algorithm 2 are as follows. First, we as domain of the reference we use $\mathcal{C}_r$ instead of $\mathcal{R}_k$. Second, we robustify (16), (17) with respect to Γ instead of $\mathcal{C}_\gamma(r)$. These modifications remove the dependency of the reference model input from the current reference model state, r. Thus, we ignore that close to the boundaries of the reference set not all the reference model inputs can actually be applied. This introduces some conservativeness in the invariant set computation. On the other hand the resulting invariant set is a polyhedron and its calculation is much faster and memory efficient. □

Remark 3: An alternative way of defining the robust control invariant set $\mathcal{C}_{x,r}$ is

$$\mathcal{C}_{x,r} \subseteq \mathcal{X}_{x,r} \tag{19a}$$

$$\begin{aligned}[x'\ r']' \in \mathcal{C}_{x,r} \Rightarrow\ \forall \gamma \in \mathcal{C}_\gamma(r)\ \exists u \in \mathcal{U} :\\ [Ax + Bu, A_r r + B_r \gamma] \in \mathcal{C}_{x,r}.\end{aligned} \tag{19b}$$

In (19) by inverting the $\forall$ and $\exists$ quantifiers compared to (12) we imply that γ_t (and thus the next reference r_{t+1}) is known to the controller at any time t. This has two implications. First, the control law κ solving Problem 1 will be a function of x_t, r_t and γ_t. Second, the set on initial states and references $\mathcal{X}_0$ where Problem 1 admits a solution will include the one obtained when using (12). However, in this case, the reference r_{t+1} implemented by the reference generator algorithm at time $t+1$ cannot differ from $r_{1|t}$ used by κ at time t, i.e., perfect preview of at least one step is needed. We are currently investigating solutions based on (19).

IV. Receding Horizon Control with Guaranteed Tracking Error Bound

The robust control invariant set $\mathcal{C}_{x,r}$ computed in Section III defines the set of states and references for which there exists an input that allow any future admissible reference signal to be tracked within the allowed error bound while enforcing system's constraints. Given the current state $x_t \in \mathcal{X}$ and reference $r_t \in \mathcal{C}_r$, there are multiple ways to compute the input $u_t \in \mathcal{U}$ such that $[x'_{t+1}\ r'_{t+1}] \in \mathcal{C}_{x,r}$. In this section we propose an approach based on model predictive control.

At time $t \in \mathbb{Z}_{0+}$, let the state $x_t \in \mathcal{X}$ and the reference state trajectory along a future horizon of length $N \in \mathbb{Z}_+$, $R_t^N = [r'_{0|t}, \ldots, r'_{N|t}]'$, be given. R_t is assumed to have been generated by the reference generator algorithm according to Assumption 1. Solve the finite horizon optimal control problem,

$$g_U(x_t, R_t^N) = \arg\min_{U_t} \sum_{k=0}^{N-1} q(y_{k|t} - y_{r,k|t}, u_{k|t}) \tag{20a}$$

$$\text{s.t.}\quad x_{k+1|t} = Ax_{k|t} + Bu_{k|t}, \tag{20b}$$

$$y_{k|t} = Cx_{k|t}, \tag{20c}$$

$$y^r_{k|t} = C_r r_{k|t}, \tag{20d}$$

$$[x'_{k|t},\ r'_{k|t}]' \in \mathcal{C}_{x,r}, \tag{20e}$$

$$u_{k|t} \in \mathcal{U}, \tag{20f}$$

$$x_{0|t} = x_t, \tag{20g}$$

$$k = 0, \ldots, N \tag{20h}$$

where $U_t = [u'_{0|t}, \ldots, u'_{N-1|t}]'$, and q is a convex stage cost Let $U_t^* = [{u^*_{0|t}}', \ldots, {u^*_{N-1|t}}']'$ be the optimal solution of (20). The first element of U_t^* is applied to system (5), hence obtaining the control law

$$g_{\mathrm{RHC}}(x_t, R_t^N) = [I\ 0 \cdots 0] g_U(x_t, R_t^N). \tag{21}$$

At time $t+1$, the optimization problem (20) is solved based on the new state $x_{0|t+1} = x_{t+1}$, and R_{t+1}^N. The next theorem presents sufficient conditions guaranteeing persistent feasibility of the MPC control law (20), (21).

Theorem 2: Consider system (5) and reference model (7) subject to constraints (6), (8), respectively, and tracking constraint (10). Let Assumption 1 be satisfied by the reference generator algorithm, and $r_{0|t+1} = r_{1|t}$ for all $t \geq 0$. Let $\mathcal{C}_{x,r}$ be a robust control invariant set for (5), (7) subject to constraints (6), (8), (10), robust to any $\gamma_r \in \mathcal{C}_\gamma(r)$, and let $u_t = g_{\mathrm{RHC}}(x_t, R_t^N)$. If (20) is feasible at time t, then (20) is feasible for all $\bar{t} \in \mathbb{Z}_{0+}$, $\bar{t} \geq t$. Moreover, constraints (6) and (10) are satisfied by the closed-loop system.

The proof follows the reasoning of classical recursive feasibility proofs of MPC, and it is omitted here due to limited space.

In Theorem 2 the role of the predicted reference trajectory $R_t^N = [r'_{0|t}, \ldots, r'_{N|t}]'$ computed by the reference generator algorithm at time t is evident. Starting from a $r_{0|t}$, the reference generator algorithm selects a feasible sequence of inputs $\gamma_{0|t}, \ldots, \gamma_{N-1|t}$ to model (7) and communicates to the tracking algorithm the sequence $R_t^N = [r'_{0|t}, r'_{1|t}, \ldots, r'_{N|t}]'$. At $t+1$, the reference generator algorithm is initialized with $r_{1|t} = r_{0|t+1}$ and the procedure is repeated. Therefore, at $t+1$ the future references $r_{1|t+1}, \ldots, r_{N-1|t+1}$ can be different from $r_{2|t}, \ldots, r_{N|t}$.

Remark 4: The results of Theorem 2 hold for any feasible controller $g_U(x_t, R_t^N)$ for problem (20), regardless of its optimality, i.e., at each step t, only a feasible solution of the optimization problem (20) is needed. □

Corollary 1: Under the assumptions of Theorem 2, controller g_{RHC} solves Problem 1 and $\mathcal{X}_0 \triangleq \mathcal{C}_{x,r}$. □

Remark 5: Consider the control invariant set $\mathcal{C}_{x,r}$, and the associated input admissible set $\mathcal{C}_u(x, r)$. If problem (20) is augmented with the additional constraint $u_{0|t} \in \mathcal{C}_u(x_{0|t}, r_{0|t})$, then the assumption $r_{0|t+1} = r_{1|t}$ can be removed from Theorem 2 and persistent feasibility is still guaranteed since $[x'_t, r'_t]' \in \mathcal{C}_{x,r}$ □

If the predicted reference along the horizon is guaranteed not to change, a slightly modified optimal control problem can be formulated. The control invariant set constraint is enforced only at the end of the prediction horizon, resulting in a larger feasible region. Details are omitted here due to limited space.

It shall be noted that in this paper we only focus on reference tracking with guaranteed error bounds, and hence we do not require steady state convergence to the reference. However, it is immediate to include an integral action [27], [28] for constant references, or use the disturbance model approach (e.g., [2], [8]) in order to obtain also steady state offset free tracking, when the reference is eventually coinciding with the output of a linear autonomous system.

V. Examples

We present an example where a second order underdamped system has to track a reference generated by a (different) second order underdamped system driven by a bounded input.

We consider the second order system

$$\dot{x} = \begin{bmatrix} -2 & -2 \\ 2 & 0 \end{bmatrix} x + \begin{bmatrix} 4 \\ 0 \end{bmatrix} u \tag{22a}$$

$$y = \begin{bmatrix} 0 & 5 \end{bmatrix} x, \tag{22b}$$

subject to constraints

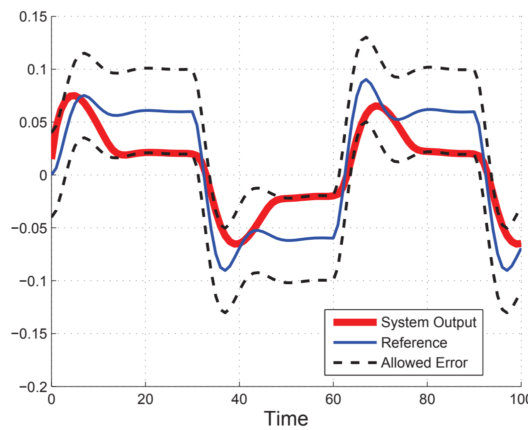
$$-0.1 \le u \le 0.1, \quad \begin{bmatrix} -0.04 \\ -0.015 \end{bmatrix} \le x \le \begin{bmatrix} 0.04 \\ 0.015 \end{bmatrix}. \tag{23}$$

We sample (22) with sampling period $T_s = 0.1$s to obtain (5). The reference model (7) is obtained by sampling with $T_s = 0.1$s

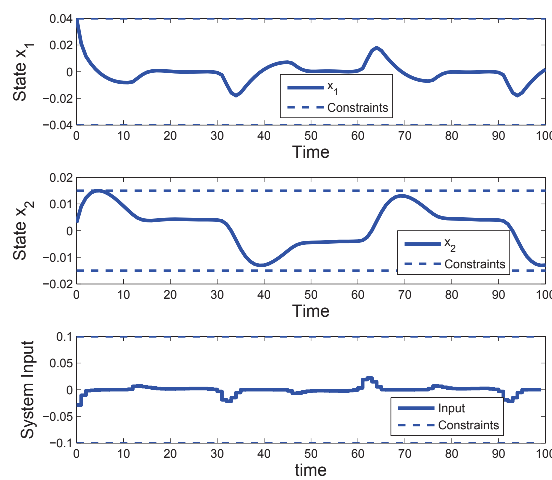
$$\dot{r} = \begin{bmatrix} -4 & -6.25 \\ 4 & 0 \end{bmatrix} r + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \gamma \tag{24a}$$

$$y_t^r = \begin{bmatrix} 0 & 3.125 \end{bmatrix} r, \tag{24b}$$

where (24) has oscillatory modes with damping ratio 0.4 and settling time 2s. In this example $\Gamma = \{\gamma \in \mathbb{R} : \ \gamma^- \le \gamma \le \gamma^+\}$, $\gamma^+ = -\gamma^- = 0.06$, and $\epsilon = 0.04$. We compute the robust control invariant $\mathcal{C}_{x,r}$ using Algorithm 2, where for $\mathcal{C}_r$ and $\mathcal{C}_\gamma(r)$ the maximal robust control invariant sets $\mathcal{C}_r^\infty$ and $\mathcal{C}_\gamma^\infty(r)$ are used, respectively. Algorithm 2 converges after 6 iterations.



(a) Reference, system output and tracking bounds



(b) State and input trajectories, and constraints

Fig. 1. Simulation of example system in closed-loop with the proposed controller.

In Figure 2 we show a closer look to part of the simulation, which highlights the complex system behavior due to the two objectives of keeping the tracking error bounded while minimizing the control effort.

We design the MPC tracking controller (20), (21) with $q(y_{k|t} - y_{r,k|t}, u_{k|t}) = u_{k|t}^2$ and prediction horizon $N = 3$.

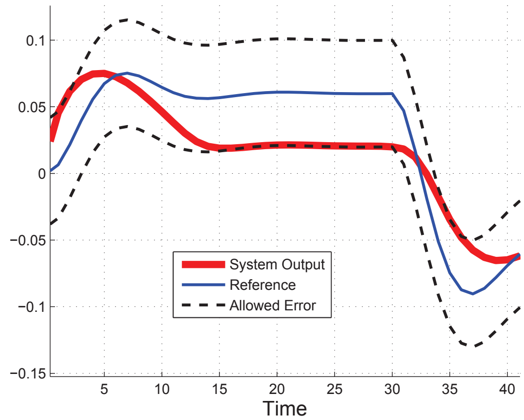


Fig. 2. Simulation of example system in closed-loop with the proposed controller. System output and reference in part of the simulation.

Figure 1 depicts the closed loop system behavior when the initial state is $x_0 = [0.04\ 0.003]'$ for a reference profile which satisfies Assumption 1.

Figure 1 shows that the synthesized control algorithm is effective in maintaining the tracking error within the desired bounds at every time step, while satisfying input and state constraints.

VI. Conclusions and Future Work

We have proposed a design technique for tracking controllers for constrained systems that guarantees satisfaction of constraints on system states and inputs, and predefined error bounds on the tracking error at every time step, for all the reference trajectories generated by a constrained linear system driven by a bounded input. The control design is based on a model predictive controller enforcing a robust control invariant set. We have proposed an algorithm to compute the robust control invariant set, and proved persistent feasibility of the MPC controller. In future works we will develop alternative formulations for the robust control invariant set and MPC controller and consider problems such as recovery an reference design.

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