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***Abstract*—This paper proposes a dynamic controller structure and a systematic design procedure for stabilizing discrete-time hybrid systems. The proposed approach is based on the concept of control Lyapunov functions (CLFs), which, when available, can be used to design a stabilizing state-feedback control law. In general, the construction of a CLF for hybrid dynamical systems involving both continuous and discrete states is extremely complicated, especially in the presence of non-trivial discrete dynamics. Therefore, we introduce the novel concept of a *hybrid control Lyapunov function*, which allows the compositional design of a discrete and a continuous part of the CLF, and we formally prove that the existence of a hybrid CLF guarantees the existence of a classical CLF. A constructive procedure is provided to synthesize a hybrid CLF, by expanding the dynamics of the hybrid system with a specific controller dynamics. We show that this synthesis procedure leads to a dynamic controller that can be implemented by a receding horizon control strategy, and that the associated optimization problem is numerically tractable for a fairly general class of hybrid systems, useful in real world applications. Compared to classical hybrid receding horizon control algorithms, the proposed approach typically requires a shorter prediction horizon to guarantee asymptotic stability of the closed-loop system, which yields a reduction of the computational burden, as illustrated through two examples.**

I. Introduction

In hybrid systems discrete dynamics, such as finite automata, Petri nets, or Markov chains, interact with continuous dynamics, such as differential, difference, or differential algebraic equations [3]–[5]. A fundamental problem in controlling hybrid dynamical systems is the stabilization of a desired hybrid equilibrium state. In the last ten years, several lines of research with different levels of generality have been devoted to this problem. These lines include, amongst others, switching control (e.g., [6]), optimal control (e.g., [7], [8]), model predictive control (e.g., [9]–[12]), control-to-facet approaches (e.g., [13], [14]), and more recently, passivity-based and CLF-based approaches (e.g., [15], [16]), see also the references therein and the surveys [4], [5], [17], [18]. However, the existing general approaches often lack a constructive nature to synthesize controllers, while the existing constructive approaches often apply only to restrictive classes of hybrid systems.

To elaborate on the latter, the majority of the constructive approaches apply to switched linear systems or piecewise affine systems, in which the discrete state (the system mode) is subordinate to the continuous state, in the sense that it is uniquely determined by the continuous state and (possibly) inputs. As such, the discrete dynamics are rather trivial. On the other hand, non-trivial discrete dynamics are essential features of various applications such as robot operation, processes control, and embedded control systems, (see, e.g., [19]–[21]), and as such, there is still a need for constructive methodologies for synthesizing stabilizing controllers for hybrid systems.

In this paper we propose a novel constructive methodology to design stabilizing controllers for general hybrid systems with non-trivial continuous and discrete dynamics. Due to the generality of the assumptions, the proposed technique is applicable to, among others, (discrete-time) hybrid automata [22], and MLD systems [9]. Due to the equivalence results in [23], [24], the proposed approach is applicable also to extended piecewise affine systems (i.e., piecewise affine systems augmented with non-trivial discrete dynamics) [24], switched linear systems [6], discrete hybrid automata [25], and many others. The design of stabilizing controllers for hybrid systems that we propose is inspired by the control Lyapunov function (CLF) approach [26], [27], where, after constructing a CLF, the synthesis of a control law that achieves stability of the controlled system equilibrium follows naturally. In general, the construction of a CLF is complex even for continuous systems, and it becomes even more complicated when considering hybrid systems.

Due to the complexity of obtaining a CLF directly, in this paper we follow a compositional approach. Instrumental in this approach is the introduction of the concept of a *hybrid CLF*, which allows for the separate design of a discrete and a continuous part of the CLF. Despite the separate design, it can be formally proven that the existence of a hybrid CLF guarantees the existence of a classical CLF in the sense of [26], [27]. We propose a constructive procedure, based on expanding the dynamics of the hybrid system with controller dynamics designed using concepts from predictive control, that leads to the systematic synthesis of the hybrid CLF. The presence of controller dynamics constitutes a further novelty, with respect to the use of classical CLFs [26] that typically are employed in

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Preliminary results related to this research were presented in [1], [2].

conjunction with static state feedback laws, instead of dynamic controllers.

Finally, we show that the control law designed via the hybrid CLF can be implemented by solving at every control cycle a finite horizon optimal control problem in a receding horizon fashion [28]. While for general hybrid systems the optimization problem arising in the receding horizon controller may be computationally challenging, for hybrid systems with affine transition guards and (piecewise) affine continuous dynamics for each discrete state, which have been proved useful in various practical applications [29]–[32], the optimization problem can be solved effectively by available numerical tools. Receding horizon control strategies for hybrid systems were proposed before, e.g., in [9]–[12]. However, previous strategies guarantee only convergence to an equilibrium for hybrid systems with discrete dynamics [9], or asymptotic stabilization (including Lyapunov stability) for piecewise affine systems with trivial discrete dynamics [10]–[12]. In order to guarantee feasibility of the optimal control problem, existing techniques require in general long horizons, due to the presence of terminal constraints. Embedding artificial candidate Lyapunov functions in optimal control problems *via* constraints (see, e.g., [33]–[35]) avoids the need of such long prediction horizons. However, recursive feasibility of the optimal control problem is not automatically guaranteed [34], unless the constraints related to the artificial candidate Lyapunov function can be proven to truly represent a CLF. For the approach proposed in this paper, recursive feasibility of the optimal control problem is guaranteed by an appropriate construction of the hybrid CLF. Thus, compared to existing hybrid receding horizon control algorithms [9]–[12], a shorter prediction horizon is usually required for the proposed design to guarantee asymptotic stability of the closed-loop system and recursive feasibility is guaranteed. Clearly the former is beneficial for the controller implementation as it yields a reduction of the computational load.

The paper is structured as follows. In Section II we briefly review the basic notions of stability, control Lyapunov functions, and some notions of graphs. In Section III we introduce the hybrid system stabilization problem, and the class of controllers that we synthesize to address it. In Section IV we show how a controller that stabilizes the hybrid system is designed by using a hybrid CLF, which is simple to obtain because of its compositional nature and is proven to guarantee the existence of a classical CLF. In Section V we propose a construction for the hybrid CLF, which guarantees the existence of the controller, and in Section VI we implement the control law via a receding horizon constrained control strategy. After presenting a numerical example and an example of launch control on mild hybrid electric vehicles, in Section VII we summarize the conclusions.

II. Preliminaries

$\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$ denote the set of real, positive real, non-negative real, integer, positive integer, and non-negative integer numbers, respectively. For a countable set $\mathcal{S}$, $|\mathcal{S}|$ denotes its cardinality. We use the notation $\mathbb{Z}_{(c_1,c_2]}$, where $c_1, c_2 \in \mathbb{Z}$, (and similarly with $\mathbb{R}$) to denote the set $\{k \in \mathbb{Z} : c_1 < k \leq c_2\}$. Given a set $\mathcal{X}$, $2^{\mathcal{X}}$ denotes the set of subsets of $\mathcal{X}$. The Hölder p-norm of a vector $x \in \mathbb{R}^n$ is defined as $\|x\|_p \triangleq (|[x]_1|^p + \ldots + |[x]_n|^p)^{\frac{1}{p}}$, if $p \in \mathbb{Z}_{[1,\infty)}$ and $\|x\|_\infty \triangleq \max_{i=1,\ldots,n} |[x]_i|$, where $[x]_i$, $i = 1, \ldots, n$, is the i-th component of x, and $|\cdot|$ is the absolute value. By $\|\cdot\|$ we denote an arbitrary p-norm, and x' denotes the transpose of x.

For a discrete-time signal $\{x(k)\}_k$, with sampling period T_s, we refer to time (instant) k as the time instant when the k^{th} sampling occurs, i.e., $t = kT_s$. Given a discrete-time system $x(k+1) = \phi(x(k), u(k))$, an initial state $x(0)$ and an input sequence $\mathbf{u}_N = \big(u_0, \ldots, u_{N-1}\big)$, $N \in \mathbb{Z}_{>0}$, $\mathbf{x}_N = \big(x_0, \ldots, x_N\big)$ is the sequence of states obtained from $x(0)$ following the application of the input sequence $\mathbf{u}_N$. For simplicity of notation, we denote $\phi^j(x(0), \mathbf{u}_N) \triangleq x_j$ for $j \in \mathbb{Z}_{[0,N]}$. For two vectors $u \in \mathbb{R}^{n_u}$ and $v \in \mathbb{R}^{n_v}$, we sometimes write $(u, v) = [u' \ v']' \in \mathbb{R}^{n_u+n_v}$. In addition, with a little abuse of notation, we sometimes separate the discrete valued and the real (continuous) valued arguments of a function $f(x, u)$, i.e., given $x = [x_c' \ x_d']'$, $u = [u_c' \ u_d']'$, where x_c, u_c are the continuous (i.e., real-valued) components, and x_d, u_d are the discrete (i.e., discrete-valued) components of x and u, respectively, we write sometimes $f(x_c, x_d, u_c, u_d) \triangleq f(x, u)$.

A. Stability notions

Consider the discrete-time nonlinear system described by the difference inclusion

$$x_c(k+1) \in \Phi_c(x_c(k)), \quad k \in \mathbb{Z}_{\geq 0}, \tag{1}$$

where $x_c(k) \in \mathbb{R}^n$ is the state at time k. The mapping $\Phi_c : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is an arbitrary nonlinear, possibly discontinuous, set-valued function. A state $x_c^e \in \mathbb{R}^n$ satisfying $\Phi_c(x_c^e) = \{x_c^e\}$ is called an equilibrium of (1). After introducing some terminology, we state a regional version of the global asymptotic stability property presented in [36, Chapter 4].

A function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. It *belongs to class* $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and $\varphi(s) \to \infty$ when $s \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *belongs to class* $\mathcal{KL}$ if for each $k \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, k) \in \mathcal{K}$, for each $s \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot)$ is decreasing, and $\lim_{k\to\infty} \beta(s, k) = 0$.

Definition 1: Consider system (1) and $\mathbb{X}_c \subseteq \mathbb{R}^n$ with $x_c^e \in \mathbb{X}_c$ and $\Phi_c(x_c^e) = \{x_c^e\}$. We call the equilibrium x_c^e *asymptotically stable (AS) in* $\mathbb{X}_c$ for (1) if there exists a $\mathcal{KL}$-function β such that, for any $x_c(0) \in \mathbb{X}_c$, all the trajectories generated by (1) satisfy

$$\|x_c(k) - x_c^e\| \leq \beta(\|x_c(0) - x_c^e\|, k), \ \forall k \in \mathbb{Z}_{>0}. \tag{2}$$

□

For systems with discrete dynamics, the discrete state domain is taken as the finite set of symbols $\mathcal{E} \triangleq \{\epsilon_1, \ldots, \epsilon_{n_d}\}$. Consider the discrete dynamical system

$$x_d(k+1) \in \Phi_d(x_d(k)), \quad k \in \mathbb{Z}_{\geq 0}, \tag{3}$$

where $x_d(k) \in \mathcal{E}$ is the state and $\Phi_d : \mathcal{E} \to 2^{\mathcal{E}}$ is an arbitrary set-valued function.

Definition 2 ([22]): Given a finite set $\mathcal{E}$ the *discrete distance* is the function $d_d : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_{\geq 0}$ given for $x_d, y_d \in \mathcal{E}$ by

$$d_d(x_d, y_d) \triangleq \begin{cases} 0 & \text{if} \quad x_d = y_d \\ 1 & \text{if} \quad x_d \neq y_d \end{cases}. \tag{4}$$

□

By using (4) we formulate an analogous version of Definition 1 for (3), for $x_d^e \in \mathcal{E}$ satisfying $\phi_d(x_d^e) = \{x_d^e\}$, i.e., x_d^e is an equilibrium of (3).

Definition 3: Consider the discrete system (3) and let $x_d^e \in \mathcal{E}$ be such that $\phi_d(x_d^e) = \{x_d^e\}$. The equilibrium x_d^e is called *asymptotically stable in* $\mathcal{E}$ for (3) if there exists a $\mathcal{KL}$-function β such that for any $x_d(0) \in \mathcal{E}$, all the trajectories generated by (3) satisfy $d_d(x_d(k), x^e) \leq \beta(d_d(x_d(0), x^e), k)$, for all $k \in \mathbb{Z}_{\geq 0}$. □

Since the set $\mathcal{E}$ is finite, Definition 3 is equivalent to the existence of $k_0 \in \mathbb{Z}_{\geq 0}$ such that for any $x_d(0) \in \mathcal{E}$, $x_d(k) = x_d^e$ for all $k \geq k_0$.

Consider now the discrete-time hybrid system given by

$$x(k+1) = \begin{bmatrix} x_c(k+1) \\ x_d(k+1) \end{bmatrix} \in \begin{bmatrix} \Phi_c(x(k)) \\ \Phi_d(x(k)) \end{bmatrix} = \Phi(x(k)), \tag{5}$$

where $x(k) = [x_c(k)' \ x_d(k)']' \in \mathbb{X} \subseteq \mathbb{X}_c \times \mathcal{E}$ is the hybrid state at time $k \in \mathbb{Z}_{\geq 0}$ with $x_c(k) \in \mathbb{X}_c \subseteq \mathbb{R}^n$ the continuous part, $x_d(k) \in \mathcal{E}$ the discrete part, $\mathcal{E}$ defined as above, and $\mathbb{X}$ the set of admissible hybrid states.

Let $x^e = [x_c^{e\prime} \ x_d^{e\prime}]' \in \mathbb{X}$, and $\Phi(x^e) = \{x^e\}$, i.e., $x^e \in \mathbb{X}$ is an equilibrium for (5). Based on Definitions 1 and 3, we define asymptotic stability for discrete-time hybrid systems that exhibit both discrete and continuous dynamics as in (5). For hybrid states $x = \left[\begin{smallmatrix} x_c \\ x_d \end{smallmatrix}\right] \in \mathbb{X}$ and $\chi = \left[\begin{smallmatrix} \chi_c \\ \chi_d \end{smallmatrix}\right] \in \mathbb{X}$, we define the distance in the hybrid state space $d_h : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_{\geq 0}$, as in [22], by

$$d_h(x, \chi) \triangleq \|x_c - \chi_c\| + d_d(x_d, \chi_d). \tag{6}$$

Definition 4: Consider hybrid system (5) and let $x^e \in \mathbb{X}$ satisfy $\Phi(x^e) = \{x^e\}$. The equilibrium x^e is called *asymptotically stable in* $\mathbb{X}$ for (5) if there exists a $\mathcal{KL}$-function β such that for any $x(0) \in \mathbb{X}$ all the trajectories generated by (5) satisfy

$$d_h(x(k), x^e) \leq \beta(d_h(x(0), x^e), k), \ \forall k \in \mathbb{Z}_{>0}. \tag{7}$$

□

Definition 4 is consistent with [22], and it coincides with Definition 1 and Definition 3 in the case of purely continuous and purely discrete systems, respectively.

B. Lyapunov functions and control Lyapunov functions

Definition 5: A set $\mathcal{P} \subseteq \mathbb{X}_c \times \mathcal{E}$ is *positively invariant (PI)* for system (5) if for all $x \in \mathcal{P}$, $\Phi(x) \subseteq \mathcal{P}$. □

Theorem 1: Let $\mathbb{X}$ be a PI set for (5) with $x^e \in \mathbb{X}$. Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathbb{R}_{[0,1)}$, and let $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be a function such that

$$\alpha_1(d_h(x, x^e)) \leq \mathcal{V}(x) \leq \alpha_2(d_h(x, x^e)) \tag{8a}$$

$$\mathcal{V}(x^+) \leq \rho \mathcal{V}(x) \tag{8b}$$

for all $x \in \mathbb{X}$, and all $x^+ \in \Phi(x)$. Then, x^e is AS for (5) in $\mathbb{X}$. □

The proof of Theorem 1 is similar in nature to the proofs in [35, Ch.6] by replacing the (continuous) difference *equation* with the hybrid difference *inclusion* (5), and it is omitted here for brevity. The proof can also be obtained by following [37], which discusses robust stability of discrete-time difference inclusions. A function $\mathcal{V}$ that satisfies the hypothesis of Theorem 1 is called a *Lyapunov function* (LF) for hybrid system (5).

Consider now the discrete-time hybrid system with control inputs described by the difference equation

$$\begin{aligned} x(k+1) &= \begin{bmatrix} x_c(k+1) \\ x_d(k+1) \end{bmatrix} = \begin{bmatrix} \phi_c(x(k), u(k)) \\ \phi_d(x(k), u(k)) \end{bmatrix} \\ &= \phi(x(k), u(k)), \end{aligned} \tag{9}$$

where $x(k) \in \mathbb{X} \subseteq \mathbb{X}_c \times \mathcal{E}$, $\mathbb{X}_c \subseteq \mathbb{R}^n$, $u(k) \in \mathbb{U} \subseteq \mathbb{U}_c \times \mathcal{E}_u$ are the state and input at $k \in \mathbb{Z}_{\geq 0}$, and $\mathcal{E}_u \triangleq \{\epsilon_{u_1}, \ldots, \epsilon_{u_{m_d}}\}$ is a finite set of input symbols. In (9) $\phi : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is an arbitrary nonlinear function, possibly discontinuous. Assume that for $x^e = [x_c^{e\prime} \ x_d^{e\prime}]' \in \mathbb{X}$ there exists $u^e = [u_c^{e\prime} \ u_d^{e\prime}]' \in \mathbb{U}$, such that $\phi(x^e, u^e) = x^e$.

Definition 6: A function $\mathcal{V}_h : \mathbb{X} \to \mathbb{R}_{\geq 0}$ that satisfies (8a) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and for which there exists $\rho \in \mathbb{R}_{[0,1)}$ such that for all $x \in \mathbb{X}$, there exists $u \in \mathbb{U}$ such that $\phi(x, u) \in \mathbb{X}$ and

$$\mathcal{V}_h(\phi(x, u)) \leq \rho \mathcal{V}_h(x), \tag{10}$$

is called a *control Lyapunov function* (CLF) for $x^e \in \mathbb{X}$ for (9). □

Given the CLF $\mathcal{V}_h$, define the control law

$$u(k) \in R(x(k)), \ k \in \mathbb{Z}_{\geq 0}, \tag{11}$$

where, for all $x \in \mathbb{X}$, $R : \mathbb{X} \to 2^{\mathbb{U}}$ satisfies

$$\emptyset \neq R(x) \subseteq \Gamma(x) := \{u \in \mathbb{U} : \phi(x, u) \in \mathbb{X}, \text{ and (10) hold}\}. \tag{12}$$

This results in the closed-loop system

$$x(k+1) \in \phi(x(k), R(x(k))) \triangleq \{\phi(x(k), u) : u \in R(x(k))\}. \tag{13}$$

Theorem 2: Consider (9) and $x^e = [x_c^{e\prime} \ x_d^{e\prime}]' \in \mathbb{X}$, where there exists $u^e \in \mathbb{U}$ such that $\phi_c(x^e, u^e) = x^e$. Suppose that there exists a CLF for x^e in $\mathbb{X}$ for (9). Then, x^e is asymptotically stable in $\mathbb{X}$ for (13). □

Theorem 2 is a consequence of Theorem 1 as $\mathbb{X}$ is PI for (13) by (12). Theorem 2 is instrumental to the main developments in this paper, since it shows that once a CLF is found, controller (11) that satisfies (12) for all $x \in \mathbb{X}$ can be constructed. If (9) consists only of continuous dynamics we obtain a classical CLF as in [27], which is the discrete-time form of the CLF for continuous-time dynamics in [26].

C. Graph notions

A directed graph $G = (V, E)$ is described by the set of *nodes* $V = \{v_1, \ldots, v_s\}$ and the set of *edges* $E \subseteq V \times V$, where $e_{ij} = (v_i, v_j) \in E$ is the edge from node $v_i \in V$ to node $v_j \in V$.

Next we introduce the notion of *graph distance*.

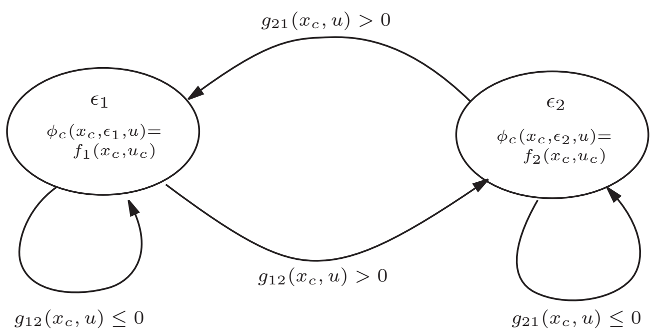


Figure 1. Graphical representation of a simple hybrid system with its associated to graph.

Definition 7: Given a graph $G = (V, E)$, a *graph path* from $v_r \in V$ to $v_t \in V$, is a sequence of vertices $\tau = (\nu^{(0)}, \dots, \nu^{(\ell)})$, $\ell \in \mathbb{Z}_{\geq 0}$, where $\nu^{(j)} \in V$ for $j \in \mathbb{Z}_{[0,\ell]}$, $(\nu^{(j)}, \nu^{(j+1)}) \in E$ for $j \in \mathbb{Z}_{[0,\ell-1]}$, and $\nu^{(0)} = v_r$, $\nu^{(\ell)} = v_t$. The length of the path is $\mathcal{L}(\tau) \triangleq \ell$, i.e., the number of edges traversed from v_r to v_s. □

For $v_r, v_t \in V$, let $\mathcal{T}_{r,t}$ denote the set of all graph paths from v_r to v_t.

Definition 8: For the directed graph $G = (V, E)$, the *graph distance* between $v_r, v_t \in V$ is the length of the shortest graph path between them, i.e., for $v_r \neq v_t$, if $\mathcal{T}_{r,t} \neq \emptyset$, $d(v_r, v_t) = \min_{\tau \in \mathcal{T}_{r,t}} \mathcal{L}(\tau)$, and if $\mathcal{T}_{r,t} = \emptyset$, $d(v_r, v_t) \triangleq \infty$. For $v_r = v_t$, $d(v_r, v_t) \triangleq 0$. □

The graph distance, which represents the minimum number of edges to travel between two nodes is a proper distance function on undirected graphs, but it lacks the symmetry property on directed graphs, since in general $d(v_r, v_t) \neq d(v_t, v_r)$. However, this does not impact our use of the graph distance. For a given graph $G(V, E)$, for all $v_r, v_t \in V$ the graph distance $d(v_r, v_t)$ can be computed using, for instance, Dijkstra's algorithm.

III. Problem Definition

Consider hybrid system (9), where, for $k \in \mathbb{Z}_{\geq 0}$, $x(k) \in \mathbb{X} \subseteq \mathbb{X}_c \times \mathcal{E}$, $\mathbb{X}_c \subseteq \mathbb{R}^n$ and $\mathcal{E} = \{\varepsilon_1, \dots, \varepsilon_{n_d}\}$, and $u(k) \in \mathbb{U} \subseteq \mathbb{U}_c \times \mathcal{E}_u$ with $\mathbb{U}_c \subseteq \mathbb{R}^m$ and $\mathcal{E}_u = \{\epsilon_{u_1}, \dots, \epsilon_{u_{m_d}}\}$. The sets $\mathbb{X}$ and $\mathbb{U}$ define the admissible sets of states and inputs, respectively, which possibly describe system constraints. Model (9) is fairly general, as it can, for instance, represent a hybrid automaton [22] (in discrete time) with control inputs and deterministic executions, see, e.g., Figure 1. While state and input constraints defined by $\mathbb{X}$ and $\mathbb{U}$ are independent from each other, this condition is introduced here only to simplify the notation, and it can be easily relaxed to allow for mixed state-input system constraints. Given $\epsilon \in \mathcal{E}$, $\mathcal{X}_h(\epsilon) \triangleq \{x \in \mathbb{X} : x_d = \epsilon\}$ is the set of hybrid states in $\mathbb{X}$ where the discrete state is ϵ, and obviously $\mathbb{X} = \bigcup_{\epsilon \in \mathcal{E}} \mathcal{X}_h(\epsilon)$, and $\mathcal{X}_h(\epsilon) = \mathcal{X}_c(\epsilon) \times \{\epsilon\}$, where $\mathcal{X}_c(\epsilon) \triangleq \{x_c \in \mathbb{R}^n : \left[\begin{smallmatrix} x_c \\ \epsilon \end{smallmatrix}\right] \in \mathbb{X}\}$ is the set of continuous states compatible with $\epsilon \in \mathcal{E}$, sometimes referred to as the *domain of* ϵ. Furthermore, given $\epsilon_i, \epsilon_j \in \mathcal{E}$, define $\mathcal{X}_t(\epsilon_i, \epsilon_j) = \{x_c \in \mathcal{X}_c(\epsilon_i) : \exists u \in \mathbb{U},\ \phi([x_c'\ \epsilon_i']', u) \in \mathcal{X}_h(\epsilon_j)\}$.

Obviously, a directed graph $G(V, E)$ can be associated to (9) in the following way. Define $V \triangleq \{v_1, \dots, v_{n_d}\}$ so that $v_i \in V$ is associated to $\epsilon_i \in \mathcal{E}$, for all $i \in \mathbb{Z}_{[1,n_d]}$. To define the set of edges, take $e_{ij} = (v_i, v_j) \in E$, for $i, j \in \mathbb{Z}_{[1,n_d]}$, if and only if $\mathcal{X}_t(\epsilon_i, \epsilon_j) \neq \emptyset$. A graphical representation of a simple hybrid system with its associated graph is shown in Figure 1. By associating $G(V, E)$ to (9) we enable the use of the graph distance for the discrete component of the hybrid system state. In fact, for the *discrete* distance (4) all the states appear equally far from a desired target state x_d^e, except x_d^e itself. Instead, the *graph* distance measures how far x_d is from x_d^e in terms of the number of discrete transitions needed to reach x_d^e.

We consider the stabilization of a desired equilibrium $x^e = [x_c^{e\prime}\ x_d^{e\prime}]' \in \mathbb{X}$, for which there exists $u^e \in \mathbb{U}$ such that $\phi(x^e, u^e) = x^e$. The general problem that this paper addresses is to provide a *constructive* design procedure to obtain a controller such that x^e is asymptotically stable for the closed-loop system in an appropriate sense. In Section II-B we have described how such a control law can be obtained from a CLF. However, the direct derivation of a CLF for (9) is far from trivial.

In order to obtain a constructive procedure to design a stabilizing controller for (9), we consider *dynamic* controllers of the type

$$z(k+1) = \psi(x(k), z(k), u(k), v(k)), \tag{14a}$$

$$\begin{bmatrix} u(k) \\ v(k) \end{bmatrix} \in R(x(k), z(k)), \tag{14b}$$

where $z \in \mathcal{Z} \subseteq \mathbb{R}^{n_z}$ is the controller state with dynamics defined by (14a), $v \in \mathbb{V}$ is an additional (endogenous) control input, and (14b) defines the set-valued command as a function of x and z. Hence, the problem addressed in this paper is formulated as follows.

Problem: Stabilizing Feedback Control Design. Given a desired equilibrium $x^e \in \mathbb{X}$ for (9) with $u^e \in \mathbb{U}$ satisfying $\phi(x^e, u^e) = x^e$, synthesize (14) such that there exist a non-trivial set $\Xi \subseteq \mathbb{X} \times \mathcal{Z}$, and $z^e \in \mathcal{Z}$, such that (x^e, z^e) is an asymptotically stable equilibrium in Ξ for the closed-loop system (9), (14). □

At a conceptual level, the approach that we take in this paper is to first appropriately select the controller dynamics (14a) such that the interconnection of (9) and (14a) allows for a CLF $\mathcal{V}_h : \Xi \to \mathbb{R}_{\geq 0}$, and then choose the feedback R such that (12) is satisfied for (9), (14). The choice of z and the construction of the dynamics (14a) are the main contributions of the paper, next to crafting the CLF in a systematic manner. The proposed approach is different from the classical CLF-based stabilization, which typically results in *static* state feedback laws, while (14) is a dynamic controller. In addition, the CLF is built in a compositional manner based on a so-called "hybrid CLF". In what follows, we first formally introduce the concept of hybrid CLF, after which we prove that if a hybrid CLF exists, then it induces a classical CLF for the hybrid system. Next, we describe a procedure based on predictive control concepts to construct the controller dynamics and the hybrid CLF. In addition, we show that the stabilizing controller can be synthesized as a receding horizon controller. Before moving to the next section, it is valuable to remark that the hybrid CLF concept that we introduce is very general, and it may allow to develop many other design procedures and control laws besides the ones proposed here.

IV. Hybrid Control Lyapunov Functions

Given a desired equilibrium $x^e \in \mathbb{X}$, we construct (14) using a so-called hybrid CLF. The hybrid CLF is shown to induce a CLF $\mathcal{V}_h$ consistent with Definition 6 for the interconnection of (9) and (14a), and it can be used for constructing R in (14b).

A. Definition of a hybrid CLF

In this section we define the concept of a hybrid CLF.

Definition 9: A hybrid CLF for system (9), (14a) for $(x^e, z^e) \in \Xi \subseteq \mathbb{X} \times \mathcal{Z}$ is a triple $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$, where $\mathcal{V}_c : \mathbb{X}_c \to \mathbb{R}_{\geq 0}$, $\mathcal{V}_d : \mathcal{E} \to \mathbb{R}_{\geq 0}$ and $\mathcal{V}_z : \mathcal{Z} \to \mathbb{R}_{\geq 0}$ satisfy the bounds

$$\alpha_1^c(\|x_c - x_c^e\|) \leq \mathcal{V}_c(x_c) \leq \alpha_2^c(\|x_c - x_c^e\|),\ \forall x_c \in \mathbb{X}_c, \quad (15a)$$

$$\alpha_1^d(d_d(x_d, x_d^e)) \leq \mathcal{V}_d(x_d) \leq \alpha_2^d(d_d(x_d, x_d^e)),\ \forall x_d \in \mathcal{E}, \quad (15b)$$

$$\alpha_1^z(\|z - z^e\|) \leq \mathcal{V}_z(z) \leq \alpha_2^z(\|z - z^e\|),\ \forall z \in \mathcal{Z} \quad (15c)$$

for some $\alpha_1^c, \alpha_2^c, \alpha_1^d, \alpha_2^d, \alpha_1^z, \alpha_2^z \in \mathcal{K}_\infty$. Moreover, for each $(x, z) \in \Xi$ there must exist $(u, v) \in \mathbb{U} \times \mathbb{V}$ such that

$$(\phi(x, u), \psi(x, z, u, v)) \in \Xi \quad (16)$$

and

$$\left\{\begin{array}{l}\mathcal{V}_c(\phi_c(x, u)) \leq \rho_c \mathcal{V}_c(x_c) + M_c \\ \mathcal{V}_z(\psi(x, z, u, v)) \leq \mathcal{V}_z(z) - 1 \\ \mathcal{V}_d(\phi_d(x, u)) \leq \mathcal{V}_d(x_d)\end{array}\right. \quad \text{if} \quad x_d \neq x_d^e \quad (17a)$$

$$\left\{\begin{array}{l}\mathcal{V}_c(\phi_c(x, u)) \leq \rho_c \mathcal{V}_c(x_c) \\ \mathcal{V}_z(\psi(x, z, u, v)) \leq \rho_z \mathcal{V}_z(z) \\ \mathcal{V}_d(\phi_d(x, u)) \leq \mathcal{V}_d(x_d)\end{array}\right. \quad \text{if} \quad x_d = x_d^e \quad (17b)$$

for some constants $\rho_c, \rho_z \in [0, 1)$, $M_c \in \mathbb{R}_{\geq 0}$.

Roughly speaking, (17) imposes $\mathcal{V}_c$ to be a local CLF for the continuous dynamics of (9) once the discrete state is equal to the desired discrete state (as in (17b)), $\mathcal{V}_z$ to be a CLF for the controller dynamics (14a), and $\mathcal{V}_d$ to be a CLF for the discrete dynamics of (9), although only non-increase is required. Next we show that the three components $\mathcal{V}_c$, $\mathcal{V}_d$, $\mathcal{V}_z$ of the hybrid CLF can be combined to obtain a classical CLF $\mathcal{V}_h$ for (9) and (14a) in the sense of Definition 6, thereby justifying the name "hybrid CLF". As constructing a classical CLF may be difficult, the hybrid CLF provides an appealing alternative, as it obtains a CLF in a compositional manner by appropriately choosing $\mathcal{V}_c$, $\mathcal{V}_d$, and $\mathcal{V}_z$, thereby providing a constructive procedure for the design of stabilizing controllers.

B. From a hybrid CLF to a CLF

In order to prove that a hybrid CLF induces a classical CLF, we need the following technical lemma.

Lemma 1: Let a hybrid CLF $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ for $(x^e, z^e) \in \Xi \subset \mathbb{X} \times \mathcal{Z}$ be given for system (9), (14a), and assume $\mathcal{Z}$ is a bounded set. Consider the function $\mathcal{V}_D : \mathcal{E} \times \mathcal{Z} \to \mathbb{R}_{\geq 0}$, $\mathcal{V}_D(x_d, z) = \mathcal{V}_d(x_d) + \mathcal{V}_z(z)$ for $(x_d, z) \in \mathcal{E} \times \mathcal{Z}$. Then, there exist $0 < \lambda_1 < 1$, and $0 < \lambda_2 < 1$ such that for all $(x, z) \in \Xi$ with $x_d \neq x_d^e$ there exists $(u, v) \in \mathbb{U} \times \mathbb{V}$ such that

$$\mathcal{V}_D(\phi_d(x, u), \psi(x, z, u, v)) \leq \lambda_1 \mathcal{V}_D(x_d, z) - \lambda_2.$$

Proof: It follows from (17a) that for $x_d \neq x_d^e$, there exists $(u, v) \in \mathbb{U} \times \mathbb{V}$ such that

$$\mathcal{V}_D(\phi_d(x, u), \psi(x, z, u, v)) \leq \mathcal{V}_z(z) - 1 + \mathcal{V}_d(x_d) = \mathcal{V}_D(x_d, z) - 1. \quad (18)$$

Define $\mathcal{V}_{D,\max} = \max\{2, \sup_{(x_d, z) \in \mathcal{E} \times \mathcal{Z}} \mathcal{V}_D(x_d, z)\}$, which is finite due to boundedness of $\mathcal{Z}$ and finiteness of $\mathcal{E}$. Take λ_1 and λ_2 such that

$$0 < 1 - \frac{1}{\mathcal{V}_{D,\max}} < \lambda_1 < 1,$$

$$0 < \lambda_2 \leq 1 - (1 - \lambda_1)\mathcal{V}_{D,\max} \leq \lambda_1 < 1.$$

Then from (18), for $x_d \neq x_d^e$ there exists (u, v) such that

$$\begin{aligned}&\mathcal{V}_D(\phi_d(x, u), \psi(x, z, u, v)) \leq \\ &\quad \mathcal{V}_D(x_d, z) - 1 = \\ &\quad \lambda_1 \mathcal{V}_D(x_d, z) + (1 - \lambda_1)\mathcal{V}_D(x_d, z) - (1 - \lambda_2) - \lambda_2 \leq \\ &\quad \lambda_1 \mathcal{V}_D(x_d, z) + (1 - \lambda_1)\mathcal{V}_{D,\max} - (1 - \lambda_2) - \lambda_2.\end{aligned}$$

Since $(1 - \lambda_1)\mathcal{V}_{D,\max} - (1 - \lambda_2) \leq 0$, $\mathcal{V}_D(\phi_d(x, u), \psi(x, z, u, v)) \leq \lambda_1 \mathcal{V}_D(x_d, z) - \lambda_2$ as claimed. ■

In Lemma 1 (and subsequent developments) $\mathcal{Z}$ is assumed to be bounded. This is in general not restrictive, as we will see later in the constructive design procedure, since the domain $\mathcal{Z}$ of the controller dynamics state and the controller dynamics ψ are design parameters.

Theorem 3: Let a hybrid CLF $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ for $(x^e, z^e) \in \Xi \subseteq \mathcal{X} \times \mathcal{Z}$ be given, and assume $\mathcal{Z}$ is bounded. Then, for a sufficiently large $\alpha > 0$, $\mathcal{V}_h : \Xi \to \mathbb{R}_{\geq 0}$, given by

$$\mathcal{V}_h(x, z) = \alpha \mathcal{V}_D(x_d, z) + \mathcal{V}_c(x_c), \quad (19)$$

where $(x, z) \in \Xi$ and $\mathcal{V}_D$ as in Lemma 1, is a CLF for (9), (14) for (x^e, z^e) in Ξ.

Proof: In this proof, for shortness we denote (x, z) by ξ and (x^e, z^e) by ξ^e. We first prove that bounds as in (8a) hold for $\mathcal{V}_h$, for any $\alpha > 0$. Without loss of generality we can consider d_h given by[1] $d_h(\xi, \xi^e) = d_d(x_d, x_d^e) + \|x_c - x_c^e\| + \|z - z^e\|$. To prove (8a) for $\mathcal{V}_h$, observe that (15) implies

$$\mathcal{V}_h(x, z) \geq \alpha\alpha_1^d(d_d(x_d, x_d^e)) + \alpha\alpha_1^z(\|z - z^e\|) + \alpha_1^c(\|x_c - x_c^e\|).$$

It is not hard to see that this yields

$$\begin{aligned}\mathcal{V}_h(x, z) \geq \quad & \min\left(\alpha\alpha_1^d\left(\frac{1}{3}d_h(\xi, \xi^e)\right), \alpha\alpha_1^z\left(\frac{1}{3}d_h(\xi, \xi^e)\right),\right. \\ & \left.\alpha_1^c\left(\frac{1}{3}d_h(\xi, \xi^e)\right)\right) = \alpha_1^h(d_h(\xi, \xi^e)),\end{aligned}$$

where α_1^h is given by $\alpha_1^h(s) = \min(\alpha\alpha_1^d(\frac{1}{3}s), \alpha\alpha_1^z(\frac{1}{3}s), \alpha_1^c(\frac{1}{3}s))$ for $s \geq 0$. Since it is

[1] Strictly speaking, $d_h(\xi, \xi^e) = d_d(x_d, x_d^e) + \|(x_c - x_c^e, z - z^e)\|$, but due to equivalence of norms in finite dimensional spaces, without loss of generality we can use $d_h(\xi, \xi^e) = d_d(x_d, x_d^e) + \|x_c - x_c^e\| + \|z - z^e\|$ for convenience.

the pointwise minimum of three $\mathcal{K}_\infty$ functions, $\alpha_1^h \in \mathcal{K}_\infty$, thereby proving the first inequality in (8a).

Similarly, due to (15),

$$\begin{aligned}&\mathcal{V}_h(x,z) \leq \\ &\quad \alpha\alpha_1^d(d_d(x_d,x_d^e)) + \alpha\alpha_1^z(\|z-z^e\|) + \alpha_1^c(\|x_c - x_c^e\|) \leq \\ &\quad\quad \alpha\alpha_1^d(d_h(\xi,\xi^e)) + \alpha\alpha_1^z(d_h(\xi,\xi^e)) + \alpha_1^c(d_h(\xi,\xi^e)) = \\ &\qquad\qquad\qquad\qquad \alpha_2^h(d_h(\xi,\xi^e)) \quad (20)\end{aligned}$$

with $\alpha_2^h(s) = \alpha\alpha_1^d(s) + \alpha\alpha_1^z(s) + \alpha_1^c(s))$, $s \geq 0$. Since it is the pointwise sum of three $\mathcal{K}_\infty$ functions, $\alpha_2^h \in \mathcal{K}_\infty$. Thus, the second inequality in (8a) is proven as well.

Due to (16), for each $\xi \in \Xi$ there exists $(u,v) \in \mathbb{U} \times \mathbb{V}$ such that $(\phi(x,u), \psi(x,z,u,v)) \in \Xi$, and (u,v) also satisfies (17). A decrease condition as in (10) can be proven for $\mathcal{V}_h$, if α is chosen such that $\alpha \geq \frac{M_c}{\lambda_2}$ with λ_2 as in Lemma 1. To show this, take $\xi \in \Xi$ with $x_d \neq x_d^e$, then for (u,v) satisfying (16)-(17a),

$$\begin{aligned}&\mathcal{V}_h(\phi(x,u), \psi(x,z,u,v)) = \\ &\quad \alpha\mathcal{V}_D(\phi_d(x,u), \psi(x,z,u,v)) + \mathcal{V}_c(\phi_c(x,u)) \leq \\ &\quad \alpha(\lambda_1\mathcal{V}_D(x_d,z) - \lambda_2) + \rho_c\mathcal{V}_c(x_c) + M_c \leq \\ &\quad \max(\lambda_1,\rho_c)(\alpha\mathcal{V}_D(x_d,z) + \mathcal{V}_c(x_c)) + M_c - \alpha\lambda_2 \leq \\ &\quad \max(\lambda_1,\rho_c)\mathcal{V}_h(x_d,z).\end{aligned}$$

For $\xi \in \Xi$ with $x_d = x_d^e$, $\mathcal{V}_D(x_d,z) = \mathcal{V}_z(z)$, and due to (17b), $\mathcal{V}_D(\phi_d(x,u), \psi(x,z,u,v)) = \mathcal{V}_z(\psi(x,z,u,v))$ for $(u,v) \in \mathbb{U} \times \mathbb{V}$ satisfying (16), (17b). Hence, for the chosen $(u,v) \in \mathbb{U} \times \mathbb{V}$

$$\begin{aligned}&\mathcal{V}_h(\phi(x,u), \psi(x,z,u,v)) = \\ &\quad \mathcal{V}_c(\phi(x,u)) + \alpha\mathcal{V}_z(\psi(x,z,u,v)) \leq \\ &\quad \rho_c\mathcal{V}_c(x_c) + \rho_z\mathcal{V}_D(x_d,z) \leq \max(\rho_c,\rho_z)\mathcal{V}_h(x,z),\end{aligned}$$

which concludes the proof. ■

From Theorem 3, the next corollary follows immediately.

Corollary 1: Let a hybrid CLF $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ for $(x^e, z^e) \in \Xi \subseteq \mathbb{X} \times \mathcal{Z}$ be given, and assume $\mathcal{Z}$ is bounded. Consider a CLF $\mathcal{V}_h$ for (9) and (14) for $(x^e, z^e) \in \Xi$ obtained as in Theorem 3 for a sufficiently large $\alpha > 0$. Then, there exists $0 \leq \rho_h < 1$ such that if $(u,v) \in \mathbb{U} \times \mathbb{V}$ satisfies (16), (17) for $(x,z) \in \Xi$, then $(u,v) \in \mathbb{U} \times \mathbb{V}$ satisfies

$$\mathcal{V}_h(\phi(x,u), \psi(x,z,u,v)) \leq \rho_h\mathcal{V}_h(x,z)) \ , \tag{21a}$$

$$(\phi(x,u), \psi(x,z,u,v)) \in \Xi \ . \tag{21b}$$

□

Corollary 1 is instrumental for designing $R : \Xi \to \mathbb{U} \times \mathbb{V}$ in (14b), since it guarantees that for $(x,z) \in \Xi$, if $(u,v) \in \mathbb{U} \times \mathbb{V}$ is chosen such that the hybrid CLF conditions (16), (17) are satisfied, the classical CLF conditions (21) are satisfied for $\mathcal{V}_h$.

C. Stabilizing dynamic controller

Due to Theorem 2 and Theorem 3, if $R : \Xi \to \mathbb{U} \times \mathbb{V}$ is chosen according to (12) for $\mathcal{V}_h$, $(x^e, z^e) \in \Xi$ is asymptotically stable for (9), (14). For $(x,z) \in \Xi$, Corollary 1 shows that if R is chosen as

$$R(x,z) := \{(u,v) \in \mathbb{U} \times \mathbb{V} \mid (16) - (17)\}, \tag{22}$$

then (12) is satisfied, and hence $(x^e, z^e) \in \Xi$ is asymptotically stable for (9), (14).

Thus, we obtain the following corollary.

Corollary 2: Let a hybrid CLF $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ for $(x^e, z^e) \in \Xi \subseteq \mathbb{X} \times \mathcal{Z}$ be given for system (9), (14a), and assume $\mathcal{Z}$ is bounded. If $R : \Xi \to \mathbb{U} \times \mathbb{V}$ is chosen as in (22), then (x^e, z^e) is asymptotically stable in Ξ for the closed-loop (9), (14). □

Next, we propose a specific design for the hybrid CLF (17) based on concepts from predictive control.

V. Construction of controller dynamics and hybrid CLF

While several different hybrid CLFs can be designed, we provide a systematic method to design (14) that stabilizes (x^e, z^e) in Ξ, based on a specific choice of the controller dynamics (14a) and of the hybrid CLF components. Then, $R : \Xi \to \mathbb{U} \times \mathbb{V}$ as in (14b) follows immediately by Corollary 2. According to this procedure, the elements that we have to select for specifying (14) through (14) are z, z^e, $\mathcal{Z}$, ψ, Ξ, $\mathcal{V}_c$, $\mathcal{V}_d$, $\mathcal{V}_z$.

A. Construction of the controller dynamics

To specify the controller dynamics (14a), and in particular z, $\mathcal{Z}$, v, $\mathbb{V}$, and ψ we exploit ideas from predictive control. Consider a desired equilibrium $x^e \in \mathbb{X}$ with equilibrium input $u^e \in \mathbb{U}$, i.e., $\phi(x^e, u^e) = x^e$. Let $\mathbf{u}_N(k) = (u_0(k), \ldots, u_{N-1}(k)) \in \mathbb{U}^N$ be a predicted input sequence at time $k \in \mathbb{Z}_{\geq 0}$ for $N \in \mathbb{Z}_{>0}$ steps in the future. Then, in (14) define $u(k) \triangleq u_0(k) \in \mathbb{U}$ and $v(k) \triangleq (u_1(k), \ldots, u_{N-1}(k)) \in \mathbb{U}^{N-1} \triangleq \mathbb{V}$. Hence, $(u(k), v(k) = \mathbf{u}_N(k)$, the predicted sequence of future inputs. Also, define the controller dynamics as

$$\psi(x,z,u,v) \triangleq \sum_{j=1}^{N} d(\phi_d^j(x,\mathbf{u}_N), x_d^e), \tag{23}$$

that is, the sum of the graph distances of the (predicted) discrete state to the equilibrium along the trajectories generated by the predicted input sequence. Hence, at time $k \in \mathbb{Z}_{\geq 0}$, the update of the controller state is

$$z(k+1) = \psi(x(k), u(k), v(k)) = \psi(x(k), \mathbf{u}_N(k)). \tag{24}$$

Equation (23) defines the next controller state $z(k+1)$ as the cumulated graph distance from step $k+1$ to $k+N$ along the predicted trajectory starting from $x(k)$ for $u(k+i) = u_i(k)$, $i \in \mathbb{Z}_{[0,N-1]}$. Note that, as common in predictive control, the actual future system trajectory is not necessarily equal to the predicted one, since at later steps the controller may choose different control actions $(u(j), v(j))$, for $j \in \mathbb{Z}_{>k}$, than the ones predicted at time k.

Note that $\mathcal{Z} \triangleq \mathbb{R}_{[0,c_z]}$, $c_z \in \mathbb{R}_{>0}$, $c_z < \infty$, as required in Theorem 3. While conditions for the selection of c_z will be discussed in details in what follows, note that by (23), $z(k) \leq N \max_{x_d \in \mathcal{E}} d(x_d, x_d^e)$, for all $k \in \mathbb{Z}_{>0}$, and hence $c_z = N \max_{x_d \in \mathcal{E}} d(x_d, x_d^e)$ is already a choice satisfying the assumption on c_z in Lemma 1.

For the subsequent discussion it is important to notice that by (24), for $z(k)$, $k \in \mathbb{Z}_{>0}$, the first element of the summation in (23) is $d(x_d(k), x_d^e)$. Hence, if $z(k) = 0$ for $k \in \mathbb{Z}_{>0}$, then $x_d(k) = x_d^e$. Thus, we take $z^e = 0$, which satisfies for $(u^e, v^e) = \mathbf{u}_N = (u^e, \ldots, u^e)$ that $\psi(x^e, z^e, u^e, v^e) = 0 = z^e$. Hence, (x^e, z^e) is the desired equilibrium for (9), (23), (24), as we already have that $\phi(x^e, u^e) = x^e$.

B. Construction of the hybrid CLF

The component $\mathcal{V}_d$ of the hybrid CLF (17) related to the discrete state of the hybrid system is defined by the discrete distance (4), i.e.,

$$\mathcal{V}_d(x_d) = d_d(x_d, x_d^e). \tag{25}$$

Thus, (17) requires that

$$d_d(\phi_d(x, u), x_d^e) \leq d_d(x_d, x_d^e). \tag{26}$$

To guarantee the feasibility of (26), we adopt the following assumption.

Assumption 1: For any $x \in \mathcal{X}_h(x_d^e)$ there exists $u \in \mathbb{U}$ such that $\phi(x, u) \in \mathcal{X}_h(x_d^e)$.

Assumption 1 requires that for any hybrid state where the discrete state is at the desired equilibrium, there exists an input that maintains it there.

The component $\mathcal{V}_z$ of the hybrid CLF is defined as

$$\mathcal{V}_z(z) = \|z\| = z, \tag{27}$$

where the second equality holds due to z being the sum of graph distances, and hence $z \geq 0$. For (24), (17) imposes that

$$\psi(x, z, \mathbf{u}_N) \leq z - 1 \quad \text{if} \quad x_d \neq x_d^e \tag{28a}$$
$$\psi(x, z, \mathbf{u}_N) \leq \quad \rho_z z \quad \text{if} \quad x_d = x_d^e, \tag{28b}$$

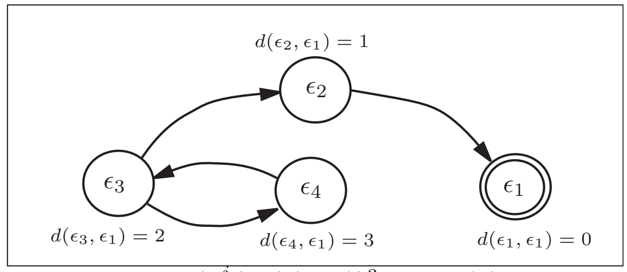
for all $(x, z) \in \Xi$ and some constant $0 \leq \rho_z < 1$. Constraint (28) is called the *cumulative graph distance contraction* constraint, and it is a relaxation of

$$d(\phi_d(x, u), x_d^e) \leq \rho_d d(x_d, x_d^e), \; 0 \leq \rho_d < 1, \tag{29}$$

that requires the discrete state to come closer to the equilibrium at every time step. Enforcing (29) is difficult and often impossible, since in most practical systems the discrete state cannot change at *every* step. In contrast, (28) requires that the sum of the graph distance along a prediction future horizon of length N to decrease when $\mathbf{u}_N$ is applied, which is a relaxed requirement. Note that if the discrete state of (9) can be controlled to approach the equilibrium at every step to enforce (29), it is possible to implement (29) by (28) with $N = 1$. Hence, (29) is equivalent to (28) for $N = 1$, while for $N > 1$ (28) is a relaxation of (29). In order to guarantee (28) we state the following assumption.

Assumption 2: Let $x^e \in \mathbb{X}$. For any discrete state $x_d \in \mathcal{E} \setminus \{x_d^e\}$ there exists $n \in \mathbb{Z}_{\geq 0}$ such that for any $x \in \mathcal{X}_h(x_d)$, there exists $\bar{x}_d \in \mathcal{E}$, where $d(\bar{x}_d, x_d^e) < d(x_d, x_d^e)$, and an input sequence $\mathbf{u}_\ell \in \mathbb{U}^\ell$, such that: (i) $\ell \leq n$; (ii) $\phi^q(x, \mathbf{u}_\ell) \in \mathbb{X}$, $\phi_d^q(x, \mathbf{u}_\ell) = x_d$, $q \in \mathbb{Z}_{[1,\ell-1]}$; (iii) $\phi_d^\ell(x, \mathbf{u}_\ell) = \bar{x}_d$. □

Definition 10: Given $x_d \in \mathcal{E}$, the *minimum graph distance horizon* $n(x_d) \in \mathbb{Z}_{\geq 0}$ for $x_d \in \mathcal{E}$ is the minimum value for which Assumption 2 holds for x_d, where $n(x_d^e) \triangleq 0$. □



k	$(\phi_d^j(x_d(k), \mathbf{u}_3))_{j=1}^3$	$\psi(k)$
0	$(\epsilon_3, \epsilon_3, \epsilon_2)$	5
1	$(\epsilon_3, \epsilon_2, \epsilon_2)$	4
2	$(\epsilon_2, \epsilon_2, \epsilon_1)$	2
3	$(\epsilon_2, \epsilon_1, \epsilon_1)$	1
4	$(\epsilon_1, \epsilon_1, \epsilon_1)$	0

Figure 2. Example of contraction of the cumulative graph distance.

Assumption 2 requires the existence of a horizon n such that from x_d, by an appropriate choice of the input sequence, a transition can be taken that brings the discrete state closer to x_d^e without leaving x_d before. As such, $n(x_d)$ is the minimum horizon needed for the discrete state to get closer to x_d^e. The value $n(x_d)$ can be computed by offline reachability analysis (see, e.g., [10], [38], [39]), as briefly discussed later in this section. The following example shows the behavior of (28a).

Example 1 (Cumulative Graph Distance Contraction): Consider the graph shown in Figure 2, where $x_d^e = \epsilon_1$. The graph distances from each node to x_d^e computed as described in Section II-C are reported in the graph close to each node. Let us assume that using the associated continuous state dynamics the following values were computed: $n(\epsilon_3) = 3$ (from ϵ_3 to ϵ_2), $n(\epsilon_2) = 2$ (from ϵ_2 to ϵ_1) and $n(\epsilon_4) = 1$ (from ϵ_4 to ϵ_3). Hence, we select $N = 3$. Given $x_d(0) = \epsilon_3$, a feasible sequence of predicted discrete state trajectories, according to the number of steps required by the underlying continuous dynamics to produce a transition of the discrete state, and the corresponding cumulative distances are given in the table in Figure 2. For comparison, the conditions in [9] for predictive control of hybrid systems, even with the relaxation in [40, Sec. 3.1], require $N \geq 5$ steps, which is the number of steps required to reach $x_d^e = \epsilon_3$. □

Remark 1: If Assumption 2 does not hold, one can still apply the proposed techniques, but the state domain has to be restricted to $\tilde{\mathbb{X}} = \bigcup_{x_d \in \mathcal{E}_v} \mathcal{X}_h(x_d)$, where $\mathcal{E}_v \subset \mathcal{E}$ is the set of the discrete states that satisfy Assumption 2. A larger subset of the state space is preserved by partitioning the domains of the discrete states in order to retain at least the parts where Assumption 2 is satisfied. □

The final component in the hybrid CLF $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ is

$$\mathcal{V}_c : \mathbb{X}_c \to \mathbb{R}_{\geq 0}, \tag{30}$$

which, by (17), should satisfy that for x with $(x, z) \in \Xi$ there exists $u \in \mathbb{U}$ such that

$$\mathcal{V}_c(\phi_c(x, u)) \leq \rho_c \mathcal{V}_c(x_c) + M_c \quad \text{if} \quad x_d \neq x_d^e \tag{31a}$$
$$\mathcal{V}_c(\phi_c(x, u)) \leq \rho_c \mathcal{V}_c(x_c) \quad \text{if} \quad x_d = x_d^e, \tag{31b}$$

where $\rho_c \in \mathbb{R}_{[0,1)}$, and $M_c \in \mathbb{R}_{>0}$. In fact, (31b) implies that $\mathcal{V}_c$ is an ordinary CLF of the continuous dynamics locally around the equilibrium x^e, and only for continuous dynamics associated to x_d^e. Finding CLFs for continuous dynamics is a well-studied problem [26], [27], and it is significantly simpler than the search for a (global) CLF for the hybrid system. Techniques for calculating local CLFs are discussed, for instance, in [34], [35], [41]. We adopt the following assumption regarding $\mathcal{V}_c$.

Assumption 3: There exists $\mathcal{V}_c$ as in (30), for which $\sup_{x\in\mathbb{X}} \mathcal{V}_c(x_c) < \infty$, and $\rho_c \in \mathbb{R}_{[0,1)}$ such that (15a) is satisfied for some $\alpha_1^c, \alpha_2^c \in \mathcal{K}_\infty$, and for all $x \in \mathcal{X}_h(x_d^e)$ there exists $u \in \mathbb{U}$ such that $\phi(x,u) \in \mathcal{X}_h(x_d^e)$ and $\mathcal{V}(\phi_c(x,u)) \leq \rho_c \mathcal{V}_c(x_c)$.

Two observations are in order. First of all, note that Assumption 3 implies Assumption 1. Second, note that to guarantee (31b) we can set $M_c = \sup_{x\in\mathbb{X}} \mathcal{V}_c(x_c)$.

Under Assumptions 2 and 3, $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ can be proven to be a hybrid CLF for (9), (23), (24).

Theorem 4: Consider (9), (23), (24), suppose Assumptions 2 and 3 hold and let $N \geq \max_{x_d\in\mathcal{E}} n(x_d)$. Define

$$\Xi = \{(x,z) \in \mathbb{X} \times \mathbb{R}_{[0,c_z]} \ : \exists (u,v) \in \mathbb{U}\times\mathbb{V}, \quad (17) \text{ holds}\}. \tag{32}$$

Then $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$, defined by (30), (25), (27), respectively, is a hybrid CLF for (9), (23), (24) for (x^e, z^e) in Ξ.

In order to prove Theorem 4 we need the following technical lemmas to prove controlled invariance of Ξ.

Lemma 2: Consider (9), (23), (24), and suppose Assumptions 1 and 2 hold. Given any $x \in \mathbb{X}$ and any $\varsigma \in \mathbb{Z}_{>0}$, there exists $\mathbf{u}_\varsigma \in \mathbb{U}^\varsigma$, such that $\phi^i(x, \mathbf{u}_\varsigma) \in \mathbb{X}$, for $i \in \mathbb{Z}_{[1,\varsigma]}$, and $d(\phi_d^{i+1}(x,\mathbf{u}_\varsigma), x_d^e) \leq d(\phi_d^i(x,\mathbf{u}_\varsigma), x_d^e)$, for $i \in \mathbb{Z}_{[0,\varsigma-1]}$.

Lemma 3: Consider (9), (23), (24), suppose Assumptions 1 and 2 hold, and let $N \geq \max_{x_d\in\mathcal{E}} n(x_d)$. Let $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ in (17) be defined by (30), (25) and (27), respectively. If (17) is satisfied for $(x,z) \in \mathbb{X}\times\mathbb{R}_{[0,c_z]}$ for some $(u,v) \in \mathbb{U}\times\mathbb{V}$, there exists $(\tilde{u},\tilde{v}) \in \mathbb{U}\times\mathbb{V}$ such that (17) is satisfied for $(\phi(x,u), \psi(x,u,v)) \in \mathbb{X}\times\mathbb{R}_{[0,c_z]}$.

The proofs of Lemma 2 and 3 are reported in Appendix A. Using Lemma 3 we can now prove Theorem 4.

Proof (Theorem 4): Given $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ defined by (30), (25) and (27), respectively, the existence of class $\mathcal{K}_\infty$ bounds on $\mathcal{V}_c$ is guaranteed by Assumption 3, while for $\mathcal{V}_d$, $\mathcal{V}_z$, it follows by construction since $\mathcal{V}_z(z) = z = \|z\|$ and $\mathcal{V}_d(x_d) = d_d(x_d, x_d^e)$. We only need to prove that for each $(x,z) \in \Xi$ there exists $(u,v) \in \mathbb{U}\times\mathbb{V}$ such that $(\phi(x,u), \psi(x,z,u,v)) \in \Xi$ and (17) is satisfied. Lemma 3 ensures that if there exists $(u,v) \in \mathbb{U}\times\mathbb{V}$, such that (17) is feasible for $(x,z) \in \mathbb{X}\times\mathbb{R}_{[0,c_z]}$, then there exists $(\tilde{u},\tilde{v}) \in \mathbb{U}\times\mathbb{V}$ such that (17) is feasible for $(\phi(x,u), \psi(x,u,v)) \in \mathbb{X}\times\mathbb{R}_{[0,c_z]}$. Hence, by choosing Ξ as in (32), for any $(x,z) \in \Xi$ there always exists $(u,v) \in \mathbb{U}\times\mathbb{V}$ such that (17) holds and $(\phi(x,u), \psi(x,z,u,v)) \in \Xi$. □

Corollary 3: Consider (9), (23), (24), let $c_z \geq N \max_{x_d\in\mathcal{E}} d(x_d, x_d^e)$, $c_z < \infty$, let Assumptions 1, 2 hold and $(\mathcal{V}_c, \mathcal{V}_d, \mathcal{V}_z)$ be defined respectively by (30), (25), (27). For any $x \in \mathbb{X}$ there exists $\bar{z} \in \mathbb{R}_{[0,c_z]}$ such that (17) is feasible for any (x,z), $z \in \mathbb{R}_{[\bar{z},c_z]}$. If (17) is feasible for $(x(0), z(0)) = (x,z)$, there exists a finite $\bar{k} \in \mathbb{Z}_{\geq 0}$ such that $z(k) = 0$, and $x_d(k) = x_d^e$, for all $k \geq \bar{k}$.

The proof of Corollary 3 is reported in Appendix A. Corollary 3 guarantees that by initializing the controller state appropriately, convergence to the equilibrium is achieved for any initial state, and that the discrete state converges in finite time to the discrete equilibrium state.

The mapping R in (22) can now be designed according to Corollary 2 providing the complete dynamical controller (14) that stabilizes (x^e, z^e) in Ξ.

Before describing a specific implementation of (22) based on receding horizon control, we discuss the imposed assumptions, their verification, and possible relaxations.

C. Verification of the Assumptions and Relaxations

The proposed technique for synthesizing hybrid CLFs is applicable to general hybrid systems, as described by (9). The restrictions in applicability are mainly due to the satisfaction of Assumptions 1 and 2, besides the existence of a local CLF which is indeed required to achieve stabilization.

Assumptions 1 and 2 are introduced to guarantee feasibility of the trajectories generated according to (17), which is needed to prove invariance of Ξ in (16). Such assumptions are always satisfied for hybrid systems that are "completely discrete-transition controllable". With reference to the graph associated to the hybrid system, this means that for every discrete state, starting from any continuous state in the associated domain and without changes in the discrete state, any outgoing (discrete state) transition may be taken in finite time, and also that the discrete state may be maintained unchanged indefinitely.

However, we would like to emphasize that complete discrete-transition controllability is a much stronger requirement than needed for Assumptions 1 and 2 to hold. These assumptions can be verified for a specific hybrid system as described next.

A value $n(x_d) \in \mathbb{Z}_{>0}$ for which Assumption 2 is satisfied for $x_d \in \mathcal{E}$ can be computed by backward reachability analysis [10], [38], [39]. Given (9) and $\mathcal{X} \subseteq \mathbb{X}$, the backward reachable set is $\mathrm{Pre}_{\phi,\mathbb{U}}(\mathcal{X}) = \{x \in \mathbb{X} : \exists u \in \mathbb{U},\ \phi(x,u) \in \mathcal{X}\}$. Consider $x_d \in \mathcal{E}$, and the set $\mathcal{E}_p(x_d) = \{\epsilon \in \mathcal{E} :\ d(\epsilon, x_d^e) < d(x_d, x_d^e)\}$, and recall that, as introduced in the beginning Section III, $\mathcal{X}_t(x_d, \epsilon)$ is the set of continuous states in the domain of x_d from which a transition to ϵ can be made. For any $\epsilon \in \mathcal{E}_p(x_d)$ define $\mathcal{S}_{x_d}^{(1)}(\epsilon) = \bar{\mathcal{S}}_{x_d}^{(1)}(\epsilon) = \{x \in \mathcal{X}_h(x_d) : x_c \in \mathcal{X}_t(x_d,\epsilon)\}$, and for $k \in \mathbb{Z}_{>0}$ compute $\mathcal{S}_{x_d}^{(k+1)}(\epsilon) = (\mathrm{Pre}_{\phi,\mathbb{U}}(\mathcal{S}_{x_d}^{(k)}(\epsilon)) \cap \mathcal{X}_h(x_d))$, $\bar{\mathcal{S}}_{x_d}^{(k+1)}(\epsilon) = \bigcup_{\ell=1}^{k+1} \mathcal{S}_{x_d}^{(\ell)}(\epsilon)$. Given $x_d^e \in \mathcal{E}$, Assumption 2 holds if and only if for any $x_d \in \mathcal{E}\backslash\{x_d^e\}$ there exists $n(x_d) \in \mathbb{Z}_{>0}$ such that $\bigcup_{\epsilon\in\mathcal{E}_p(x_d)} \bar{\mathcal{S}}_{x_d}^{(n(x_d))}(\epsilon) \supseteq \mathcal{X}_h(x_d)$.

Similarly, Assumption 1 is satisfied if and only if $\mathrm{Pre}_{\phi,\mathbb{U}}(\mathcal{X}_h(x_d^e)) \supseteq \mathcal{X}_h(x_d^e)$.

All the reachable set calculations are simplified by the fact that the discrete state has to remain constant, and hence multiple separate reachability computations involving only ϕ_c and $\mathcal{X}_t(x_d, \epsilon)$ (for constant x_d) are performed. The reachable sets can be computed exactly or by conservative approximations, depending on ϕ_c and $\mathcal{X}_t(x_d, \epsilon)$.

When compared to the assumption in [9] for predictive control of hybrid systems, that is N-steps controllability to the equilibrium for every $x \in \mathbb{X}$, and that only guarantees convergence, Assumption 2 is usually easier to verify due to its local nature, as opposed to the global nature of the assumption in [9].However, there are some cases where it is possible to find $N \in \mathbb{Z}_{>0}$ such that the assumption in [9] holds, while Assumption 2 cannot be satisfied for any $n(x_d) \in \mathbb{Z}_{>0}$. On the other hand, in most of the cases, if Assumption 2 can be satisfied, $\max_{x_d} n(x_d)$ is considerably smaller than the prediction horizon $N \in \mathbb{Z}_{>0}$ needed in [9]. In fact, Assumption 1 requires controllability to a discrete state closer to the equilibrium, rather than controllability to the equilibrium. The potentially restricting nature of Assumption 2 compared to [9] is that the discrete state has to remain constant until a discrete state closer to the target is reached.

However, Assumption 2 can be further relaxed as follows.

Assumption 4: Let $x^e \in \mathbb{X}$. For any discrete state $x_d \in \mathcal{E} \setminus \{x_d^e\}$ there exists $n \in \mathbb{Z}_{\geq 0}$ such that for any $x \in \mathcal{X}_h(x_d)$, there exists $\bar{x}_d \in \mathcal{E}$, where $d(\bar{x}_d, x_d^e) < d(x_d, x_d^e)$, and an input sequence $\mathbf{u}_\ell \in \mathbb{U}^\ell$, such that: (i) $\ell \leq n$; (ii) $\phi^q(x, \mathbf{u}_\ell) \in \mathbb{X}$, $d(\phi_d^q(x, \mathbf{u}_\ell), x_d^e) = d(x_d, x_d^e)$, $q \in \mathbb{Z}_{[1,\ell-1]}$; (iii) $\phi_d^\ell(x, \mathbf{u}_\ell) = \bar{x}_d$. □

In Assumption 4 it is only requested that the discrete state distance does not increase before decreasing at the end of the control sequence, i.e., the discrete state can change as long as the graph distance does not increase. Using Assumption 4 instead of Assumption 2 causes only minor changes in the proofs of Lemma 2 and 3, and the main claims still hold. Note though that, verifying Assumption 4 by backward reachability analysis is more involved, since the discrete state is not necessarily constant, and hence the entire hybrid system dynamics (given by ϕ_c and ϕ_d) are involved in the computations.

A further relaxation of Assumption 2 can be obtained by simply requiring existence of an N-steps trajectory from any $x \in \mathbb{X}$, such that the cumulative graph distance along the trajectory is less than $Nd(x_d, x_d^e)$, and requiring Assumption 1 to hold for every $x_d \in \mathcal{E}$ rather than only for x_d^e. This relaxation makes the assumptions closer to the ones adopted in [9], but, as for [9], the procedure for verifying it becomes more involved. Finally, as already mentioned in Remark 1, further relaxation is possible by splitting a discrete state (and its corresponding domain) into multiple discrete states.

VI. Implementation of the Stabilizing Dynamic Controller

The hybrid CLF satisfying (17) and synthesized as described in Section V results in a controller (14) with R as in (22) that generates a predicted input sequence along a future horizon. Hence, it is natural to implement R by receding horizon control. While for general hybrid systems the computation of the receding horizon control may be complex, for the classes of hybrid systems in [9], [23]–[25] where for each discrete state the continuous state dynamics are (piecewise) affine, the receding horizon control enforcing (17) can be computed efficiently by available numerical algorithms.

A. Implementation by receding horizon control

Corollary 3 guarantees that for any $x \in \mathbb{X}$ there exists a finite value $\bar{z} \in \mathbb{R}_{\geq 0}$ such that $(x, \bar{z}) \in \Xi$. Hence for any arbitrary $x \in \mathbb{X}$, with an appropriate initialization of the controller state z, $(x, z) \in \Xi$. Thus, for any initial state, Corollary 2 guarantees convergence to the desired equilibrium based on any control law that satisfies (17), i.e., for any $(u, v) \in R(x, z)$ with R as in (22). The actual input $(u, v) \in R(x, z)$ can be chosen by optimizing a performance criterion over the set of admissible inputs. In this way a receding horizon control strategy based on the repetitive solution of an optimization problem is obtained. A common definition of the performance criterion in optimization-based receding horizon control, such as model predictive control is

$$J(x, \mathbf{u}_N) \triangleq F(\phi^N(x, \mathbf{u}_N)) + \sum_{h=0}^{N-1} L(\phi^h(x, \mathbf{u}_N), u_h), \quad (33)$$

where $F : \mathbb{X} \to \mathbb{R}_{\geq 0}$ and $L : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_{\geq 0}$ denote suitable terminal and stage costs, respectively. Cost (33) typically trades off the regulation performance, in terms of distance from the equilibrium, and the actuation effort. The fixed parameters ρ_c and M_c in (17) can also be substituted by optimization variables, which may result in an improved convergence rate for the continuous state. Let ρ_c be the constant in (31), choose $\bar{\rho}_c \in \mathbb{R}_{[\rho_c, 1)}$, and let $\rho \in \mathbb{R}_{[0, \bar{\rho}_c]}$ and $M \in \mathbb{R}_{>0}$ be two additional variables that play the role of ρ_c and M_c, respectively. Instead of (31), for $x \in \mathbb{X}$ enforce

$$\mathcal{V}_c(\phi_c^1(x, \mathbf{u}_N)) \leq \rho \mathcal{V}_c(x_c) + M d_d(x_d, x_d^e), \quad (34)$$

and consider the modified cost function

$$J(x, \mathbf{u}_N, M, \rho) \triangleq w_\rho \rho + w_M M + F(\phi^N(x, \mathbf{u}_N)) + \sum_{h=0}^{N-1} L(\phi^h(x, \mathbf{u}_N), u_h), \quad (35)$$

in which the term $w_\rho \rho$, where $w_\rho \in \mathbb{R}_{>0}$, optimizes the decay of the local CLF, while $w_M M$, where $w_M \in \mathbb{R}_{>0}$, penalizes the relaxation of (34) when $d_d(x_d, x_d^e) > 0$. Whenever $M = 0$, the continuous state evolves satisfying (8b), independently of the current value of x_d. However, for stability this is required only when $x_d = x_d^e$.

Constraint (28) can be implemented by a single constraint as

$$\mathcal{V}_z(\psi(x(k), \mathbf{u}_N(k))) \leq (1 - d_d(x_d(k), x_d^e))\rho_z \mathcal{V}_z(z(k)) - d_d(x_d(k), x_d^e).$$

Algorithm 1: (Hybrid CLF Receding Horizon Control)
Initialization. Set $k = 0$, measure $x(0) \in \mathbb{X}$ and set $z(0) \geq \bar{z}$, where for $\bar{z} \in \mathbb{R}_{\geq 0}$, $(x(0), \bar{z}) \in \Xi$.

Step 1. Solve the optimization problem

$$\min_{\mathbf{u}_N(k),M(k),\rho(k)} \quad J(x(k),\mathbf{u}_N(k),M(k),\rho(k)) \tag{36a}$$
$$\text{s.t.}: \quad x_{h+1} = \phi(x_h,u_h(k)), \tag{36b}$$
$$z_1 = \psi(x_0,\mathbf{u}_N(k)) \tag{36c}$$
$$\mathcal{V}_c(\phi_c^1(x_0,\mathbf{u}_N(k))) \le \rho(k)\mathcal{V}_c(x_{0c}(k)) + M(k)d_d(x_{0d},x_d^e) \tag{36d}$$
$$\mathcal{V}_z(\psi(x_0,\mathbf{u}_N(k))) \le (1-d_d(x_{0d}(k),x_d^e))\rho_z\mathcal{V}_z(z_0) - d_d(x_{0d},x_d^e) \tag{36e}$$
$$\mathcal{V}_d(\phi_d^1(x_0,\mathbf{u}_N(k))) \le \mathcal{V}_d(x_{0d}) \tag{36f}$$
$$\mathbf{u}_N(k)\in\mathbb{U}^N,\ x_h\in\mathbb{X},\ h\in\mathbb{Z}_{[1,N]} \tag{36g}$$
$$0\le M(k)<M_c,\ 0\le\rho(k)\le\bar{\rho}_c \tag{36h}$$
$$x_0=x(k),\ z_0=z(k). \tag{36i}$$

Step 2. Let $\bar{\mathbf{u}}_N(k)=(\bar{u}_0(k),\dots,\bar{u}_{N-1}(k))$ be a feasible solution of (36), possibly, but not necessarily, an optimal one. Set $u(k)=\bar{u}_0(k)$, and $z(k+1)=\psi(x(k),\bar{\mathbf{u}}_N(k))$.
Step 3 Measure $x(k+1)$, set $k\leftarrow k+1$, and go to Step 1.

Algorithm 1 enforces the hybrid CLF via constraints and minimizes the performance criterion (35). Due to the choice of the horizon N, and Assumption 2, any $z(0)\in\mathbb{R}_{\ge 0}$ such that $z(0)\ge Nd(x_d(0),x_d^e)$ guarantees $z(0)\ge\bar{z}$. Finally, by choosing $z(0)=\max_{x_d\in\mathcal{E}} d(x_d,x_d^e)$, $(x,z(0))\in\Xi$, for all $x\in\mathbb{X}$.

Theorem 5: Given a hybrid system where dynamics (9) and the sets $\mathbb{X}$, $\mathbb{U}$ can be formulated by piecewise affine update equations and linear inequalities on real and integer variables, respectively, and F, L are linear or convex quadratic functions of their arguments, for ψ, $\mathcal{V}_z$, $\mathcal{V}_d$ specified as in Section V, and $\mathcal{V}_c(x)=\|P(x_c-x_c^e)\|_\infty$, where $P\in\mathbb{R}^{p\times n_c}$ has full column rank, (36) can be formulated as a mixed integer linear or quadratic program (MILP/MIQP). □

Proof: Represent the elements of $\mathcal{E}$ by the unitary vectors of $\mathbb{R}^{n_d}$, $\mathbb{E}_{n_d}=\{\varpi_j\}_{j=1}^{n_d}$, where $n_d=|\mathcal{E}|$. Thus, ϵ_j is represented by the j^{th} unitary vector of $\mathbb{R}^{n_d}$, ϖ_j, the vector entirely composed of 0, except for the j^{th} coordinate, which is 1. As a result, the discrete state of the hybrid system is encoded by Boolean vectors in a *one-hot encoding*, i.e., $x_d\in\mathbb{R}^{n_d}$, $[x_d]_i\in\{0,1\}$ and $\sum_i[x_d]_i=1$. Repeat the operation for the discrete input so that $u_d\in\mathbb{E}_{m_d}$, where $m_d=|\mathcal{E}_u|$.

Construct $\Delta_{x_d}=\sum_{j=0}^{n_d}\varpi_j-\varpi^e$, where $\varpi^e\in\mathbb{E}_{n_d}$ is the unitary vector representing x_d^e, and, for any $x_d\in\mathbb{E}_{n_d}$, define

$$\mathcal{V}_d(x_d)=d_d(x_d,x_d^e)=\Delta_{x_d}'x_d, \tag{37}$$

where by construction $\Delta_{x_d}'x_d=1$ iff $x_d\ne x_d^e$, and $\Delta_{x_d}'x_d=0$ iff $x_d=x_d^e$. Also, for a given x_d^e, for all $x_d\in\mathcal{E}$, we have

$$d(x_d,x_d^e)=D_{x_d^e}'x_d, \tag{38}$$

where $D_{x_d^e}\in\mathbb{Z}_{\ge 0}^{n_d}$ is a vector with i^{th} component equal to the graph distance from $x_d=\epsilon_i$ to x_d^e, i.e., $[D_{x_d^e}]_i=d(\varpi_i,x_d^e)$.

Using (37), (36f) is enforced by

$$\Delta_{x_d}'\phi_d(x_0,u_0)\le\Delta_{x_d}'x_{0d}. \tag{39}$$

By (38), for given $\mathbf{u}_N(k)\in\mathbb{U}^N$ and $x(k)\in\mathbb{X}$, (36c) is formulated as

$$z_1=\sum_{h=1}^{N}D_{x_d^e}'\phi_d^h(x_0,\mathbf{u}_N(k)). \tag{40}$$

Thus, constraint (36e) is enforced by

$$\sum_{h=1}^{N}D_{x_d^e}'\phi_d^h(x_0,\mathbf{u}_N(k))\le(1-\Delta_{x_d}'x_{0d})\rho_z z_0-\Delta_{x_d}'x_{0d}. \tag{41}$$

Constraint (36d) is formulated as

$$\|P(\phi_c(x_{0c},u_0(k))-x_c^e)\|_\infty \le \rho(k)\|P(x_{0c}-x_c^e)\|_\infty+M\Delta_{x_d}'x_{0d}. \tag{42}$$

Since (35) is linear in M and ρ, and by assumption F, L are linear or convex quadratic functions, (36a) is convex and linear or quadratic. By (42), considering that $x(k)$, $z(k)$, and hence $\mathcal{V}_c(x(k))$, are fixed at each $k\in\mathbb{Z}_{\ge 0}$, inequality (36d) is linear in M, ρ. Due to the assumptions, the results in [9], and (40), (36b) and (36c) can be formulated by mixed-integer linear inequalities in $\mathbf{u}_N(k)$. The left-hand side of (42) admits a formulation which is linear in $\mathbf{u}_N(k)$, see for instance [35]. Due to (39) and (41), (36e) and (36f) are mixed-integer linear in $\mathbf{u}_N(k)$. Constraints (36g) are mixed-integer linear in $\mathbf{u}_N(k)$ by assumption, and (36h) is linear in $\rho(k)$ and $M(k)$. Thus, (36) can be formulated as a mixed integer linear or quadratic program. ■

Under the assumptions of Theorem 5, the receding horizon control problem (36) can be formulated as a MILP/MIQP with a convex real relaxation, for which a global optimizer can be found in finite time [42]. The assumptions of Theorem 5 are satisfied by several classes of hybrid systems that have been proved useful in real applications [30]–[32], including the MLD systems [9], [25], and all the equivalent classes of hybrid systems [23], [24]. The classes of functions L, F that satisfy the assumptions of Theorem 5 include, the weighted 1 and ∞ norms and squared 2-norms of state and input vectors.

Algorithm 1 has also advantageous numerical properties which may also be exploited for future efficient implementations in more general classes of hybrid systems.

In hybrid receding horizon control, the complexity of the optimization problem depends combinatorially on the number of (discrete) variables, whose number increases linearly with the horizon length. In [9], the horizon N has to be long enough to guarantee controllability to the equilibrium *state* within N steps, while in [35] it must be long enough to guarantee controllability within N steps to a terminal *set* containing the equilibrium. In the proposed hybrid CLF-based approach, the horizon N must only guarantee controllability to a discrete state that is closer to the target than the current one. Usually, when Assumption 2 is satisfied, the horizon N of the approach proposed here is significantly shorter than the horizon needed by [9], [12]. The type of the optimization problem remains the same as in [9], [12], since only additional mixed-integer linear inequalities are involved. Thus, in general, the CLF-based approach presented in this paper requires a shorter horizon and hence the solution of a simpler optimization problem, as far as the stabilization of the equilibrium is the main concern.

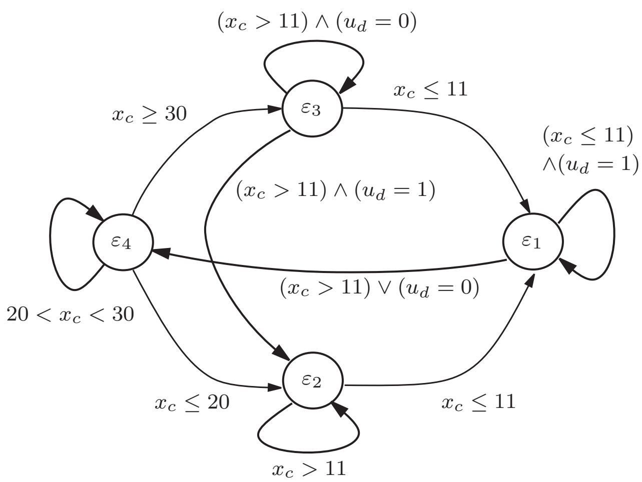
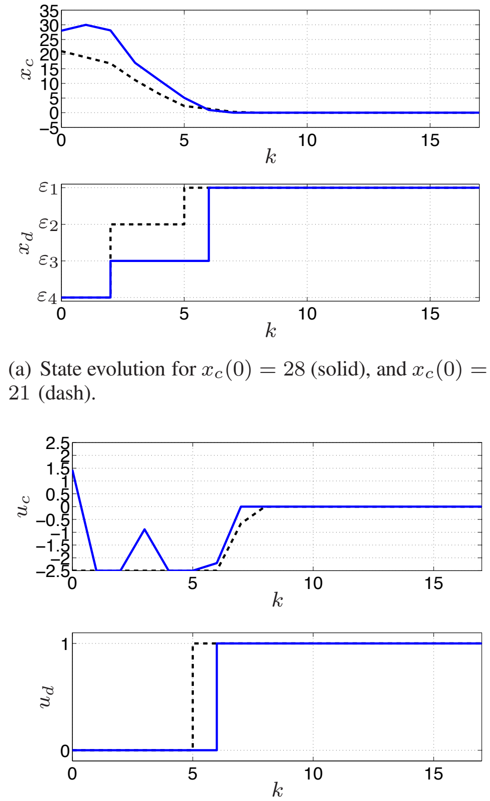
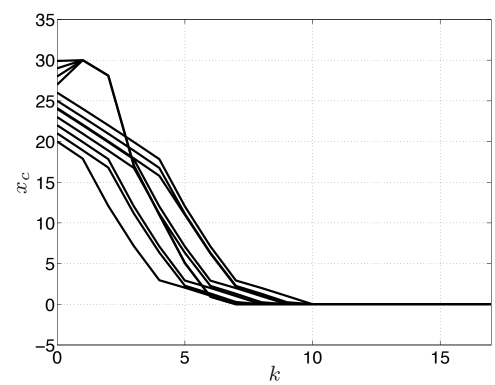


Figure 3. Graph and transition conditions associated to the discrete dynamics of the hybrid system in the numerical example.



(a) State evolution for $x_c(0) = 28$ (solid), and $x_c(0) = 21$ (dash).

(b) Input evolution for $x_c(0) = 28$ (solid), and $x_c(0) = 21$ (dash).



(c) State evolution from different initial conditions.

Figure 4. Simulation results for the numerical example.

Also, any feasible solution of (36) enforces the hybrid CLF conditions (17), according to Corollary 2. Thus, it is not necessary to attain the (global) optimum of (36) for closed-loop stability, but only to obtain a feasible solution. Hence, in Algorithm 1, the calculation of (36) can terminate as soon as a feasible solution is found. This is also potentially useful for future implementation in hybrid systems with nonlinear continuous state dynamics. In such cases (36) results in a mixed integer nonlinear programming (MINLP) problem, for which finding the global optimum is challenging. In fact, the continuous relaxations of MINLP are nonlinear programming (NLP) problems, for which it is known that the corresponding algorithms find only, in general, local optima or feasible solutions. However, for our approach a feasible solution still guarantees asymptotic stability, thereby resulting in reduced requirements for the solution of the NLP relaxations, and, as a consequence, the overall MINLP solution may be simplified. Still, the efficient solution of (36) for nonlinear dynamics poses interesting challenges that will be subject of future developments. Next we present examples of the application of the proposed techniques to systems that satisfy the assumptions of Theorem 5.

B. Numerical example

We consider a system with one continuous state, $x_c \in [-5, 30]$, four discrete states $x_d \in \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, one continuous input $u_c \in [-2.5, 2.5]$ and one discrete input $u_d \in \{0, 1\}$. Hence, $\mathbb{U} = [-2.5, 2.5] \times \{0, 1\}$, and $\mathbb{X} \subseteq [-5, 30] \times \{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, where in particular $\mathcal{X}_c(\epsilon_1) = [-5, 11.1]$. The graph and the transition conditions for the discrete dynamics of the hybrid system in the example are shown in Figure 3. The continuous dynamics are $x(k+1) = A_i x(k) + B_i u(k)$, if $x_d = \epsilon_i$, where $(A_1, B_1) = (1.07, 0.4)$, $(A_2, B_2) = (0.85, 1.25)$, $(A_3, B_3) = (0.7, 1.05)$, $(A_4, B_4) = (1.02, 1)$.

The desired equilibrium is $x_c^e = 0$, $x_d^e = \epsilon_1$ for steady state input $u_c^e = 0$, $u_d^e = 1$. The controller cost is $L(x, u) = \|Q_x(x - x^e)\|_\infty + \|Q_u(u - u^e)\|_\infty$, $Q_x = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $Q_u = \left[\begin{smallmatrix} 0.1 & 0 \\ 0 & 0.1 \end{smallmatrix}\right]$, the horizon is $N = 4$, which satisfies Assumption 2, and $\mathcal{V}_c(x_c) = \|x_c\|_\infty$, $\bar{\rho}_c = 0.98$ define the local CLF for the continuous state. The hybrid system was formulated as a discrete hybrid automaton [25], and problem (36) was formulated as a mixed-integer linear program.

Figure 4 shows the simulation results. The dash lines show the simulation results for the case $x(0) = [21 \;\; \epsilon_1]'$, where (8b) is satisfied at every step. The simulation results for the case when $x(0) = [28 \;\; \epsilon_1]'$ are shown by solid lines, where $\mathcal{V}_c$ is not monotonically decreasing along the whole trajectory. This is according to (31), where the decrease of $\mathcal{V}_c$ is required only in the set $\mathcal{X}_c(\epsilon_1)$.

It is worth to point out that for the same setup, the optimization problem formulated as in [9] is infeasible unless a longer horizon (at least $N = 9$) is used.

C. Mild HEV launch control example

We consider a problem in controlling a Hybrid Electric Vehicle (HEV) powertrain [43]. The most common configuration of HEV powertrain in today's passenger vehicles is the powersplit configuration, also called parallel-series, where the traction sources are the engine and an electric motor. Through a planetary gear, the powerflow can circulate in all directions between these, e.g., directly from the motor and/or the engine to the wheels, or from the engine to the battery, or from the engine to the motor and back to the wheels, and any combinations of these. As a consequence, the powertrain operates in different modes, including: electric motor (EM) mode, where only the electric motor drives the wheels; internal combustion engine (IC) mode, similar to a standard vehicle; positive split (PS), where the engine and the motor both provide power to the wheels; and negative split (NS), where the engine powers the wheel while the motor drains power to recharge the battery. Due to the need of enforcing several complex operating conditions, the switching logic between these modes is often implemented via finite state machines with transitions triggered by the vehicle and powertrain dynamics. Hence, the HEV powertrain is controlled by a hybrid control system.

While designing the entire HEV control strategy is out of the scope of this paper, we present the application of the control algorithm developed here to a specific prototypical problem related to launch control on a mild-HEV, where the term "mild" indicates that a small battery is used. In this operation the vehicle is accelerated from very low speed, where it is running in EM mode, to high speed, here 31m/s (approximately 70mph), where it is running in IC mode. The initial battery state of charge (SoC) is in an interval around the setpoint, and the final battery state of charge has to be at the setpoint, to accommodate future launches. The vehicle can be accelerated by negative split, thus with a low acceleration but recharging the battery, or by positive split first, thus with high acceleration and discharging the battery, and then by negative split. In IC mode, the battery is slowly charged/discharged to the setpoint, while traction is provided mainly by the combustion engine.

Let $i \in \{IC, EM, PS, NS\}$ be the mode index. For $i \in \{EM, PS, NS\}$, the equations describing the system dynamics are

$$v_v(k+1) = v_v(k) + \frac{T_s}{m}(\gamma_i u(k) - \beta v_v(k) - F_r) \quad (43a)$$
$$soc(k+1) = soc(k) - T_s \sigma_i u(k) \quad (43b)$$

where $v_v \in [0, 40]$m/s is the vehicle velocity, $soc \in [-20, 20]\%$ is the state of charge of the battery in percentage, with 0 being the charge setpoint, $u \in [0, 1]$ is the (normalized) amount of the available tractive force in the current mode that is fed to the wheels, m[kg] is the vehicle mass, β and F_r are parameters that represent an affine resistance force model (rolling resistance, bearing friction, and linearized airdrag),

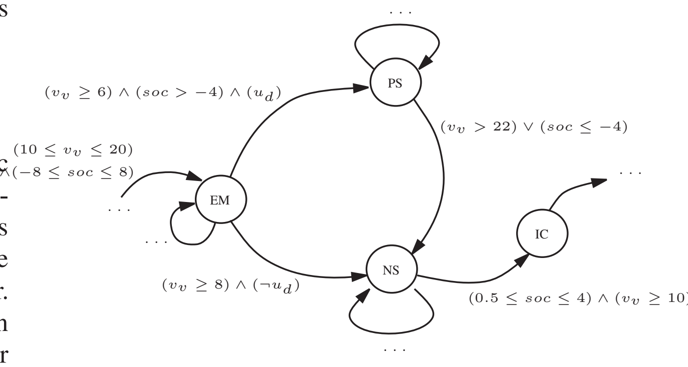


Figure 5. Automaton describing the HEV launch control logics.

and $\sigma_i[\%/s]$, γ_i[N], for $i \in \{EM, PS, NS\}$, are mode-dependent parameters. For $i = IC$, the dynamics are

$$v_v(k+1) = v_v(k) + \frac{T_s}{m}(\gamma_i u + \varrho soc(k) - \beta v_v(k) - F_r) \quad (44a)$$
$$soc(k+1) = \eta soc(k) \quad (44b)$$

where η and ϱ[N/%] are known parameters. Due to the different modes we have $\gamma_{\text{EM}} < \gamma_{\text{NS}} < \gamma_{\text{IC}} < \gamma_{PS}$. In PS mode a larger tractive force is available and the battery is discharged ($\sigma_{PS} > 0$), while in NS mode the battery is recharged ($\sigma_{NS} < 0$), but a smaller tractive force is available. The graph and the conditions on the transitions associated to the discrete dynamics are reported[2] in Figure 5. Besides conditions on the continuous states, a discrete input controls the transition from EM and PS and from PS to NS. Note that for this example, Assumption 4 provides an effective relaxation of Assumption 2, since discrete states PS and EM have equal graph distance from the desired equilibrium (IC). Hence, by Assumption 4, the horizon N can be selected considering *also* trajectories that switch between EM and PS, before reaching NS. The overall system can be represented as a Discrete Hybrid Automaton [25], which is a subclass of (9).

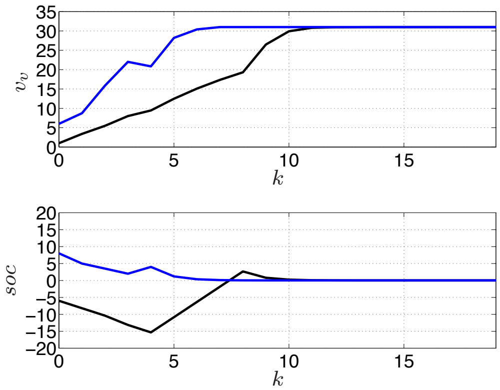
Starting from EM mode and $v_v \in [1, 8]$m/s, $soc \in [-8, 8]\%$ we want to stabilize the system on $x_c^e = [31\ 0]'$ and $x_d^e = IC$. Basing on reachability analysis, we have implemented control Algorithm 1 with horizon $N = 5$, and stage cost

$$L(x, u) = \|Q_x(x - x^e)\|_\infty + \|Q_u(u - u^e)\|_\infty,$$

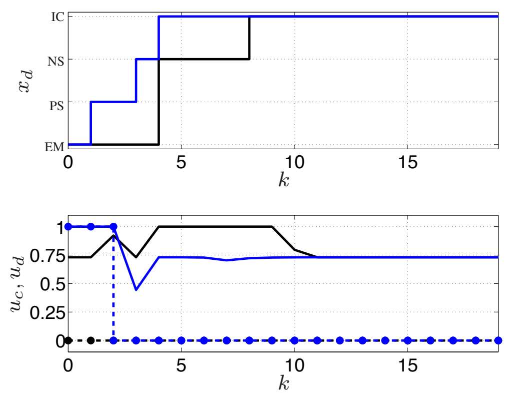
where u^e is the equilibrium input corresponding to x_c^e while in IC mode.

Simulations for different initial conditions are reported in Figure 6. Even though the PS mode graph distance is not smaller than the one of EM mode, the controller may go through it to take advantage of the high acceleration, as long as the cumulative graph distance along the horizon decreases. Thus, depending on the initial velocity and state of charge, the controller may decide to take or not to take advantage of the PS mode, and in all the cases it stabilizes the desired equilibrium. For comparison, a classical hybrid receding horizon control

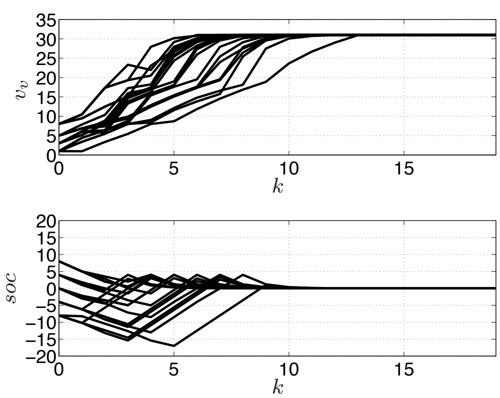
[2]The "looping transitions" guards are the complement of the outgoing transitions, and are not shown for simplicity.



(a) State evolution for $x_c(0) = [6\ 8]'$ (black), and $x_c(0) = [1\ -6]'$ (blue).



(b) Input evolution for $x_c(0) = [6\ 8]'$ (black), and $x_c(0) = [1\ -6]'$ (blue).



(c) State evolution from different initial conditions.

Figure 6. Simulation results for the mild HEV launch control system.

based on terminal equality constraints, needs a horizon of at least $N = 15$ to yield a feasible optimization problem for all the initial conditions[3] for numerical robustness.

VII. Conclusions

We have proposed a constructive method to design dynamic controllers that asymptotically stabilize the equilibrium of hybrid systems exhibiting both continuous and discrete dynamics. The key idea is to introduce a hybrid control Lyapunov function, which is simple to construct due to its compositional nature and guarantees the existence of a classical control Lyapunov function, thereby enabling a systematic design of stabilizing controllers. In fact, we have described a specific design procedure for constructing the hybrid CLF and the stabilizing dynamic controller based on predictive control concepts. We have demonstrated that the proposed control law can be implemented by receding horizon control. The optimization problem associated to such receding horizon control for various cases of interest is formulated as a MILP/MIQP, and has advantageous numerical properties such as a shorter prediction horizon than current approaches, and stability guaranteed by any feasible solution of the optimization problem. Inspired by such properties, future works will be devoted to finding efficient numerical algorithms for solving the underlying optimization problems that arise in applying the approach to more general classes of hybrid systems.

[3]In standard hybrid MPC [9], we have relaxed the terminal constraint on soc into the (small) terminal set $soc \in [-0.25, 0.25]\%$.

Appendix A
Technical Proofs

Proof of Lemma 2

Consider the case $x \in \mathcal{X}_h(x_d^e)$. By Assumption 1 there exists $u \in \mathbb{U}$ such that $\phi(x,u) \in \mathbb{X}$, and $\phi_d(x,u) = x_d^e$, hence $d(\phi_d(x,u), x_d^e) = d(x_d, x_d^e)$. Consider the case $x \notin \mathcal{X}_h(x_d^e)$. By Definition 8 and Assumption 2 there exists an input sequence $\mathbf{u}_\ell \in \mathbb{U}^\ell$ such that $\phi^i(x,\mathbf{u}_\ell) \in \mathbb{X}$, for $i \in \mathbb{Z}_{[0,\ell]}$, $\phi_d^i(x,\mathbf{u}_\ell) = x_d$, for $i \in \mathbb{Z}_{[0,\ell-1]}$, and $d(\phi_d^\ell(x,\mathbf{u}_\ell), x_d^e) < d(x_d, x_d^e)$. By iterating either of the cases above, input sequences of arbitrary length $\varsigma \in \mathbb{Z}_{>0}$ can be constructed, thus proving the lemma.

Proof of Lemma 3

We prove that if (17) is feasible for $(x,z) \in \mathbb{X} \times \mathbb{R}_{[0,c_z]}$ by $(u,v) = \mathbf{u}_N = (u_0, \ldots, u_{N-1}) \in \mathbb{U}^N$, there exists $(\tilde{u},\tilde{v}) = \tilde{\mathbf{u}}_N = (\tilde{u}_0, \ldots, \tilde{u}_{N-1}) \in \mathbb{U}^N$ such that (17) is feasible for $(\tilde{x},\tilde{z}) \triangleq (\phi(x,u), \psi(x,u,v))$. Note that conditions (17) translate into (26), (28), (31) for the particular choice of $\mathcal{V}_c$, $\mathcal{V}_d$, $\mathcal{V}_z$ proposed in Section V.

We consider the two cases, $\tilde{x}_d = x_d^e$ and $\tilde{x}_d \neq x_d^e$. First, let $\tilde{x}_d = x_d^e$. Assumption 3 guarantees that there exists $\tilde{\mathbf{u}}_N \in \mathbb{U}^N$ such that for all $i \in \mathbb{Z}_{[1,N]}$, $\phi^i(\tilde{x},\tilde{\mathbf{u}}_N) \in \mathbb{X}$, $\phi_d^i(\tilde{x},\tilde{\mathbf{u}}_N) = x_d^e$, and (26), (31) hold. In addition, for such choice of $\tilde{\mathbf{u}}_N$, $\psi(\tilde{x},\tilde{u},\tilde{v}) = 0$ is feasible, which means that (28) is feasible for any $\rho_z \geq 0$. Hence, (17) is satisfied by such choice.

Now, let $\tilde{x}_d \neq x_d^e$. Then, (26) certainly holds since $d_d(x_d, x_d^e) \leq 1$ for all $x \in \mathbb{X}$, and (31) holds by the choice of $M_c = \sup_{x\in\mathbb{X}} \mathcal{V}_c(x_c)$. Thus, we only need to prove that (28a) holds.

Let $\mathcal{J} \subseteq \mathbb{Z}_{[2,N]}$, where $j \in \mathcal{J}$ if $d(\phi_d^j(x,\mathbf{u}_N), x_d^e) < d(\phi_d^1(x,\mathbf{u}_N), x_d^e)$, and consider the two sub-cases, $\mathcal{J} \neq \emptyset$ and $\mathcal{J} = \emptyset$. Let $\mathcal{J} \neq \emptyset$, and $\bar{\jmath} = \min_{j\in\mathcal{J}} j$. Then, by Assumption 2 there exists $\bar{\mathbf{u}}_{N-\bar{\jmath}} = (\bar{u}_0, \ldots, \bar{u}_{N-\bar{\jmath}-1}) \in \mathbb{U}^{N-\bar{\jmath}}$ such that for $\tilde{\mathbf{u}}_N = (u_1, \ldots, u_{\bar{\jmath}}, \bar{\mathbf{u}}_{N-\bar{\jmath}}) \in \mathbb{U}^N$, $\phi^i(\tilde{x},\tilde{\mathbf{u}}_N) \in \mathbb{X}$, for all

$i \in \mathbb{Z}_{[0,N]}$, and

$$d(\phi_d^i(\tilde{x}, \tilde{\mathbf{u}}_N), x_d^e) = d(\phi_d^{i+1}(x, \mathbf{u}_N), x_d^e),\ i \in \mathbb{Z}_{[0,\bar{\jmath}-1]},$$
$$d(\phi_d^i(\tilde{x}, \tilde{\mathbf{u}}_N), x_d^e) \leq d(\phi_d^{\bar{\jmath}}(x, \tilde{\mathbf{u}}_N), x_d^e),\ i \in \mathbb{Z}_{[\bar{\jmath},N]}.$$

Hence,

$$\mathcal{V}_z(\psi(\tilde{x}, \tilde{u}, \tilde{v})) - \mathcal{V}_z(z) \leq$$
$$d(\phi_d^{\bar{\jmath}}(\tilde{x}, \tilde{\mathbf{u}}_N), x_d^e) - d(\phi_d^1(x, \mathbf{u}_N), x_d^e) \leq -1.$$

Let $\mathcal{J} = \emptyset$, i.e., for all $j \in \mathbb{Z}_{[2,N]}$, $\phi_d^j(x, \mathbf{u}_N) \geq \phi_d^1(x, \mathbf{u}_N)$, so that $\tilde{z} \geq Nd(\phi_d^1(x, \mathbf{u}_N), x_d^e)$. By the choice of N and Assumption 2, there exists an input sequence $\hat{\mathbf{u}}_{\bar{\jmath}} = (\hat{u}_0, \ldots, \hat{u}_{\bar{\jmath}-1}) \in \mathbb{U}^{\bar{\jmath}}$, $\bar{\jmath} \leq N$, such that $\phi^j(\tilde{x}, \hat{\mathbf{u}}_{\bar{\jmath}}) \in \mathbb{X}$, for all $j \in \mathbb{Z}_{[0,\bar{\jmath}]}$, and

$$d(\phi_d^i(\tilde{x}, \hat{\mathbf{u}}_{\bar{\jmath}}), x_d^e) = d(\tilde{x}, x_d^e),\ \forall i \in \mathbb{Z}_{[1,\bar{\jmath}-1]} \quad (45a)$$
$$d(\phi_d^{\bar{\jmath}}(\tilde{x}, \hat{\mathbf{u}}_{\bar{\jmath}}), x_d^e) < d(\tilde{x}, x_d^e). \quad (45b)$$

Then, let $x_{\bar{\jmath}} = \phi^{\bar{\jmath}}(\tilde{x}, \hat{\mathbf{u}}_{\bar{\jmath}})$. By Lemma 2 there exists $\breve{\mathbf{u}}_{N-\bar{\jmath}} = (\breve{u}_0, \ldots, \breve{u}_{N-\bar{\jmath}-1}) \in \mathbb{U}^{N-\bar{\jmath}}$ such that $\phi^j(x_{\bar{\jmath}}, \breve{\mathbf{u}}_{N-\bar{\jmath}}) \in \mathbb{X}$, for all $j \in \mathbb{Z}_{[0,N-\bar{\jmath}]}$, and

$$d(\phi_d^j(x_{\bar{\jmath}}, \breve{\mathbf{u}}_{N-\bar{\jmath}}), x_d^e) \leq d(x_{\bar{\jmath}}, x_d^e),\ \forall j \in \mathbb{Z}_{[1,N-\bar{\jmath}]}. \quad (46)$$

By choosing $\tilde{\mathbf{u}}_N = (\hat{\mathbf{u}}_{\bar{\jmath}}, \breve{\mathbf{u}}_{N-\bar{\jmath}})$ we have that

$$\psi(\tilde{x}, \tilde{u}, \tilde{v}) \leq (\bar{\jmath} - 1)d(\phi_d^1(x, \mathbf{u}_N), x_d^e) +$$
$$d(\phi_d^{\bar{\jmath}}(\tilde{x}, \tilde{\mathbf{u}}_N), x_d^e) + \sum_{N-\bar{\jmath}}^{N} d(\phi_d^j(\tilde{x}, \tilde{\mathbf{u}}_N), x_d^e)$$

and because of (45), (46),

$$\mathcal{V}_z(\psi(\tilde{x}, \tilde{u}, \tilde{v})) - \mathcal{V}_z(z) \leq$$
$$(N-\bar{\jmath}+1)\Big(d(\phi_d^{\bar{\jmath}}(\tilde{x}, \hat{\mathbf{u}}_{\bar{\jmath}}), x_d^e) - d(\phi_d^1(x, \mathbf{u}_1), x_d^e)\Big) \leq -(N-\bar{\jmath}+1)$$

which satisfies (28a) since $\bar{\jmath} \leq N$.

□

Proof of Corollary 3

Given any $x \in \mathbb{X}$, by choosing $\bar{z} = Nd(x_d, x_d^e) \leq c_z < \infty$, (17) is feasible for any (x, z), where $z \geq \bar{z}$, since (26), (31) are feasible by Assumption 3, and (28) is feasible by the choice of z and Theorem 4. Hence, for any $x \in \mathbb{X}$, if $z \geq \bar{z}$, $(x, z) \in \Xi$.

Next, we prove that there exists $\bar{k}$ such that from $(x(0), z(0)) = (x, z)$, $x_d(k) = x_d^e$, $z(k) = 0$, for all $k \geq \bar{k}$. First notice that by (24), for any $k \geq 1$, $z(k) \in \mathbb{Z}_{\geq 0}$. Thus, if $z(k) \neq 0$, $\Delta z(k) \triangleq z(k+1) - z(k) \leq -1$ regardless[4] of $x_d(k)$.

Let us assume that $z(k) > 0$ for all $k \in \mathbb{Z}_{\geq 0}$. Then,

$$z(k+1) = z(0) + \sum_{j=0}^{k} \Delta z(j) \leq z(0) - (k+1)$$

Thus, $\lim_{k\to\infty} z(k) \leq \lim_{k\to\infty} z(0) - k - 1 = -\infty$. However, we have assumed that $z(k) > 0$ for all $k \in \mathbb{Z}_{\geq 0}$, and we have reached contradiction. Thus, there must exist $\bar{k}$ such that $z(k) = 0$ for all $k \geq \bar{k}$. This also implies that $x_d(k) = x_d^e$, for all $k \geq \bar{k}$.

□

[4]When $x_d(k) \neq x_d^e$, by (28), $z(k+1) < z(k)$ and since $z(k), z(k+1) \in \mathbb{Z}_{\geq 0}$, $z(k+1) \leq z(k) - 1$.

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