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**Abstract**

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# Numerical Decomposition of Symmetric Linear Systems

Claus Danielson and Stefan Bauer

***Abstract*— This paper proposes a method for numerically decomposing symmetric linear systems. We define system symmetries as transformations of the inputs, outputs, and states that do not change the system behavior. We show that symmetric systems can be decomposed into decoupled subsystems. We provide an algorithm for performing this decomposition that uses the input-output symmetries and minimal realizations to calculate the decomposition.**

## I. Introduction

Symmetries are transformations of a system's inputs, outputs, and states for which the system is invariant. Symmetry can be used to decompose a system into decoupled subsystems. This property has been exploited in several applications. In [1] it was shown that controllability and stability in large scale systems can be determined by checking the smaller decoupled subsystems. In [2] the authors used this decomposition to simplify the design of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controllers by designing the controllers for each of the decoupled subsystems separately. In [3] symmetry was used to eliminate symmetrically redundant portions of the controller. In general, once the system has been decoupled, any of the control method presented in [4] can be applied. While many papers have focused on the benefits of exploiting symmetry, none have addressed the issue of numerically computing the symmetric decomposition. This is the issue addressed in this paper.

In recent years the semi-definite programming community has addressed the closely related problem of decomposing matrix $*$-algebras [5], [6], [7]. The set of all square matrices with a particular symmetry group forms a matrix $*$-algebra. The proposed algorithms for decomposing matrix $*$-algebras work by computing the spectral decomposition of a properly chosen element of the matrix $*$-algebra. The decomposition of this single matrix can be used to construct the decomposition of every matrix in the set. Although the theory behind these algorithms is complex, the algorithms are simple and can be implemented using existing matrix decomposition tools. In this paper we show how to apply these algorithms for the decomposition of symmetric linear systems. Further information about the use of symmetry in optimization can be found in [8], [9], [10].

One limitation of the decomposition algorithms found in the literature, is that they require a basis for the matrix $*$-algebra. In general computing the basis of the matrix $*$-algebra from the symmetry group can be difficult. One exception is when the elements of the symmetry group are permutation matrices. In many interesting applications, the input-output symmetries are indeed permutations. For instance battery networks [11], HVAC [12], paper-manufacturing [13], multi-agent systems [14], and distributed control [15]. However the nature of the state-space symmetries depends on the realization of the system. In this paper we study the relationship between input-output and state-space symmetries. We provide a method for computing the state-space symmetries from the input-output symmetries and provide a condition for when the state-space symmetries are permutation matrices. The main result of this paper is that we can decompose a linear system without the state-space symmetries by decomposing its transfer function and computing a minimal realization.

## II. The Structural Theorem

In this section we introduce the notation and mathematical concepts used in this paper. In particular we state the Structural Theorem [16] on which this paper is based.

The set of finite-dimensional, causal, rational transfer functions will be denoted by $\mathcal{R}$.

For a matrix $G \in \mathbb{C}^{n\times n}$ the subspace $\mathbb{V} \subseteq \mathbb{C}^n$ is called invariant if $G\mathbb{V} \subseteq \mathbb{V}$. For a set of matrices $\mathcal{G} \subseteq \mathbb{C}^{n\times n}$ the subspace $\mathbb{V} \subseteq \mathbb{C}^n$ is invariant if $\mathbb{V}$ is invariant $G\mathbb{V} \subseteq \mathbb{V}$ for every matrix $G \in \mathcal{G}$ in the set $\mathcal{G}$. For any matrix $G \in \mathbb{C}^{n\times n}$ the zero-space $\mathbb{O} = \mathrm{Span}(0)$ and the full-space $\mathbb{C}^n$ are called the trivially invariant subspaces. An invariant subspace $\mathbb{V} \subseteq \mathbb{C}^n$ is called irreducible if its only invariant subspaces are trivial. For a set of matrices $\mathcal{G} \subseteq \mathbb{C}^{n\times n}$ an invariant subspace $\mathbb{V} \subseteq \mathbb{C}^n$ is irreducible if its only invariant subspaces are trivial.

The direct-sum $G_1 \oplus G_2$ of matrices $G_1 \in \mathbb{C}^{n_1\times m_1}$ and $G_2 \in \mathbb{C}^{n_2\times m_2}$ is the matrix $G_1 \oplus G_2 = \left[\begin{smallmatrix} G_1 & 0 \\ 0 & G_2 \end{smallmatrix}\right] \in \mathbb{C}^{(n_1+n_2)\times(m_1+m_2)}$. For matrix sets $\mathcal{G}_1 \subseteq \mathbb{C}^{n_1\times m_1}$ and $\mathcal{G}_2 \subseteq \mathbb{C}^{n_2\times m_2}$ their direct-sum $\mathcal{G}_1 \oplus \mathcal{G}_2 \;=\subseteq\; \mathbb{C}^{(n_1+n_2)\times(m_1+m_2)}$ is the set $\mathcal{G}_1 \oplus \mathcal{G}_2 = \left\{ \left[\begin{smallmatrix} G_1 & 0 \\ 0 & G_2 \end{smallmatrix}\right] : G_1 \in \mathcal{G}_1 \text{ and } G_2 \in \mathcal{G}_2 \right\}$.

The Kronecker product of matrices $G \in \mathbb{C}^{n\times m}$ and $H \in \mathbb{C}^{p\times q}$ is the matrix

$$G \otimes H = \begin{bmatrix} g_{11}H & \dots & g_{1n}H \\ \vdots & \ddots & \vdots \\ g_{1m}H & \dots & g_{nm}H \end{bmatrix} \in \mathbb{C}^{np\times mq}.$$

If $I \in \mathbb{C}^{r\times r}$ is the identity matrix then the Kronecker product $I_r \otimes G = \bigoplus_{i=1}^r G \in \mathbb{C}^{rn\times rm}$ is the the direct-sum of the matrix $G \in \mathbb{C}^{n\times m}$ with itself $r$ times.

### A. The Structural Theorem

A group $(\mathcal{H}, \circ)$ is a set $\mathcal{H}$ along with a binary operator $\circ$ such that the operator is associative, the set $\mathcal{H}$ is closed under the operation $\circ$, contains an identity element and the inverse of each element. A representation $\Theta : \mathcal{H} \to \mathbb{C}^{n\times n}$ is a set of matrices $\Theta(g)$ index by $g \in \mathcal{H}$ that obey the group multiplication law $\Theta(g)\Theta(h) = \Theta(g\circ h)$ for all $g, h \in \mathcal{H}$. For notational simplicity we will drop the $\circ$ and write $gh$ for $g \circ h$.

Let $\Theta_1(g) \in \mathbb{C}^{n_1\times n_1}$ and $\Theta_2(g) \in \mathbb{C}^{n_2\times n_2}$ be two representations of a group $\mathcal{H}$. We define the commutator-space $\mathcal{H}(\Theta_1, \Theta_2)$ as the set of all matrices $G \in \mathbb{C}^{n_1\times n_2}$ that commute with $\Theta_1(g)$ and $\Theta_2(g)$

$$\mathcal{H}(\Theta_1, \Theta_2) = \left\{ G \in \mathbb{C}^{n_1\times n_2} : \Theta_1(g)G = G\Theta_2(g), \forall g \in \mathcal{H} \right\}.$$

This matrix set $\mathcal{H}(\Theta_1, \Theta_2)$ is a vector-subspace of the matrix space $\mathbb{C}^{n_1\times n_2}$ i.e. for every $G_1, G_2 \in \mathcal{H}(\Theta_1, \Theta_2)$ and $\alpha_1, \alpha_2 \in \mathbb{C}$ we have $\alpha_1 G_1 + \alpha_2 G_2 \in \mathcal{H}(\Theta_1, \Theta_2)$. Since $\mathcal{H}(\Theta_1, \Theta_2)$ is a vector-space we can find a set of basis-matrices $G_1, \dots, G_m$ that span this space $\mathcal{H}(\Theta_1, \Theta_2) = \mathrm{Span}(\{G_1, \dots, G_m\})$. This fact alone has many applications. However the commutator-space $\mathcal{H}(\Theta_1, \Theta_2)$ has remarkable structural properties summarized by the following Theorem.

*Theorem 1 (Structural):* Let $\mathcal{H}$ be a finite group. The commutator-space $\mathcal{H}(\Theta_1, \Theta_2) \subseteq \mathbb{C}^{n_1\times n_2}$ is the direct sum of vector-spaces $\mathbb{C}^{n_1^i \times n_2^i}$

$$\mathcal{H}(\Theta_1, \Theta_2) = \bigoplus_{i=1}^{p} I_{r_i} \otimes \mathbb{C}^{n_1^i \times n_2^i} \tag{1}$$

where $n_1 = \sum_{i=1}^{p} r_i n_1^i$ and $n_2 = \sum_{i=1}^{p} r_i n_2^i$

*Proof:* See [16] ■

This theorem says that every matrix $G \in \mathbb{C}^{n_1 \times n_2}$ in the commutator $\mathcal{H}(\Theta_1, \Theta_2) \subseteq \mathbb{C}^{n_1 \times n_2}$ can be decomposed into $p$ smaller matrices $G_1 \in \mathbb{C}^{n_1^1 \times n_2^1}, \dots, G_p \in \mathbb{C}^{n_1^p \times n_2^p}$ each with repetition $r_i$. Furthermore any matrix $G \in \mathbb{C}^{n_1 \times n_2}$ with the decomposition (1) is an element of the commutator $\mathcal{H}(\Theta_1, \Theta_2) \subseteq \mathbb{C}^{n_1 \times n_2}$.

According to the Structural Theorem 1, there exists a basis of $\mathbb{C}^{n_1}$ and $\mathbb{C}^{n_2}$, called the symmetry adapted basis, such that every element of the commutator $\mathcal{H}(\Theta_1, \Theta_2)$ is block-diagonal. This decomposition is analogous to the singular value decomposition of a single matrix, but for the set $\mathcal{H}(\Theta_1, \Theta_2)$ of matrices. There exists orthogonal transformation matrices

$$\Phi_1 = \left[\Phi_1^{11}, \dots, \Phi_1^{1r_1} | \quad \dots \quad |\Phi_1^{p1}, \dots, \Phi_1^{pr_p}\right] \in \mathbb{C}^{n_1 \times n_1}$$
$$\Phi_2 = \left[\Phi_2^{11}, \dots, \Phi_2^{1r_1} | \quad \dots \quad |\Phi_2^{p1}, \dots, \Phi_2^{pr_p}\right] \in \mathbb{C}^{n_2 \times n_2}$$

that block-diagonalize every element $G \in \mathcal{H}(\Theta_1, \Theta_2)$ of the commutator $\mathcal{H}(\Theta_1, \Theta_2) \subseteq \mathbb{C}^{n_1 \times n_2}$

$$\Phi_1^{-1} G \Phi_2 = \begin{bmatrix} I_{r_1} \otimes G_{11} & & \\ & \ddots & \\ & & I_{r_p} \otimes G_{pp} \end{bmatrix}$$

where $\otimes$ is the Kronecker product. The submatrices $\Phi_1^i = \left[\Phi_1^{i1}, \dots, \Phi_1^{ir_i}\right] \in \mathbb{C}^{n_1 \times r_i n_1^i}$ and $\Phi_2^i = \left[\Phi_2^{i1}, \dots, \Phi_2^{ir_i}\right] \in \mathbb{C}^{n_2 \times r_i n_2^i}$ decompose the matrix $G \in \mathcal{H}(\Theta_1, \Theta_2)$ into blocks $(\Phi_1^i)^* G (\Phi_2^j) = I_{r_i} \otimes G_{ii} \in \mathbb{C}^{r_i n_1^i \times r_i n_2^i}$. The submatrices $\Phi_1^{ij} \in \mathbb{C}^{n_1 \times n_1^i}$ and $\Phi_2^{ij} \in \mathbb{C}^{n_2 \times n_2^i}$ produce the $r_i$ copies of the block $G_{ii} = (\Phi_1^{ij})^* G (\Phi_2^{ij}) \in \mathbb{C}^{n_1^i \times n_2^i}$. For any $j, k = 1, \dots, r_i$ the blocks $(\Phi_1^{ij})^* G \Phi_2^{ij}$ and $(\Phi_1^{ik})^* G \Phi_2^{ik}$ are identical $(\Phi_1^{ij})^* G (\Phi_2^{ij}) = (\Phi_1^{ik})^* G (\Phi_2^{ik})$.

The columns of $\Phi_1^{ij}$ and $\Phi_2^{ij}$ span the irreducible invariant subspaces $\mathbb{V}_1^{ij} = \mathrm{Span}(\Phi_1^{ij})$ and $\mathbb{V}_2^{ij} = \mathrm{Span}(\Phi_1^{ij})$ of every matrix $G \in \mathcal{H}(\Theta_1, \Theta_2)$. These invariant subspaces $\mathbb{V}_1^{ij}$ and $\mathbb{V}_2^{ij}$ are unique. However the choice of basis vectors $\Phi_1^{ij}$ and $\Phi_2^{ij}$ for these invariant subspaces $\mathbb{V}_1^{ij}$ and $\mathbb{V}_2^{ij}$ are not unique. Thus the symmetry adapted basis are not unique. We can produce another set of symmetry adapted basis using "local" similarity transformations $T_1^i \in \mathbb{C}^{n_1^i \times n_1^i}$ and $T_2^i \in \mathbb{C}^{n_2^i \times n_2^i}$ for $i = 1, \dots, p$

$$\Phi_1 = \left[T_1^1 \Phi_1^{11}, \dots, T_1^1 \Phi_1^{1r_1} | \quad \dots \quad |T_1^p \Phi_1^{p1}, \dots, T_1^p \Phi_1^{pr_p}\right]$$
$$\Phi_2 = \left[T_2^1 \Phi_2^{11}, \dots, T_2^1 \Phi_2^{1r_1} | \quad \dots \quad |T_2^p \Phi_2^{p1}, \dots, T_2^p \Phi_2^{pr_p}\right].$$

An alternative version of the Structural Theorem 1 for decomposing real-valued commutator-spaces $\mathcal{H}(\Theta_1, \Theta_2) \subseteq \mathbb{R}^{n_1 \times n_2}$ over the real numbers $\mathbb{R}$ can be found in [7]. In this case the matrix set $\mathcal{H}(\Theta_1, \Theta_2)$ does not necessarily decompose into arbitrary real-value matrices $\mathbb{R}^{n_1^i \times n_2^i}$. Instead the matrix set $\mathcal{H}(\Theta_1, \Theta_2) \subseteq \mathbb{R}^{n_1 \times n_2}$ can decompose into real represents of complex or quaternion matrices. In this paper we will primarily consider the simpler case of decomposing complex matrices into complex matrices.

## III. Symmetric Linear Systems

In this section we define input-output and state-space symmetries, and show that they are equivalent.

An input-output symmetry is a transformation of the inputs $\Theta_u$ and outputs $\Theta_y$ that does not change the input-output behavior of the system $G(s)$.

*Definition 1:* A (spatial) symmetry of the transfer function matrix $G(s) \in \mathcal{R}^{n_y \times n_u}$ is a pair of invertible matrices $\Theta_y \in \mathbb{C}^{n_y \times n_y}$ and $\Theta_u \in \mathbb{C}^{n_u \times n_u}$ such that

$$\Theta_y G(s) = G(s) \Theta_u \tag{2}$$

for all $s \in \mathbb{C}$.

This definition says that the systems $G(s)$ and $\Theta_y^{-1} G(s) \Theta_u$ have identical input-output behavior. Equivalently this definition says that the response $y(t)$ of the systems $G(s)$ to the input $u(t)$ is related to the response $\Theta_y y(t)$ of the system to the input $\Theta_u u(t)$.

The set of all symmetries $\Theta_y$ and $\Theta_u$ of the transfer function matrix $G(s)$ form an infinite group denoted by $\mathrm{Aut}(G)$.

*Proposition 1:* The set $\mathrm{Aut}(G)$ of all symmetries $\Theta_y \in \mathbb{C}^{n_y \times n_y}$ and $\Theta_u \in \mathbb{C}^{n_u \times n_u}$ of the transfer function matrix $G(s) \in \mathcal{R}^{n_y \times n_u}$ is an infinite group.

*Proof:* See [17]. ■

The input $\Theta_u$ and output $\Theta_y$ transformations are representations of the abstract group $\mathcal{H} = \mathrm{Aut}(G)$. Thus we will index the pairs $\Theta_u(g)$ and $\Theta_y(g)$ for $g \in \mathcal{H} = \mathrm{Aut}(G)$.

A state-space symmetry is a transformation of the input-space $\Theta_u$, output-space $\Theta_y$, and state-space $\Theta_x$ that preserve the state-space matrices $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.

*Definition 2:* A symmetry of the state-space $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a triple of invertible matrices $\Theta_y \in \mathbb{C}^{n_y \times n_y}$, $\Theta_u \in \mathbb{C}^{n_u \times n_u}$, and $\Theta_x \in \mathbb{C}^{n_x \times n_x}$ such that

$$\begin{bmatrix} \Theta_x & 0 \\ 0 & \Theta_y \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Theta_x & 0 \\ 0 & \Theta_u \end{bmatrix}. \tag{3}$$

This definition says that the state-spaces matrices $A$, $B$, $C$, and $D$ that describe the system are unchanged by the transformations $\Theta_{x,u,y}$. This definition makes a statement about the system data $A$, $B$, $C$, and $D$ rather than the system behavior.

The set of all symmetries $\Theta_y$, $\Theta_u$, and $\Theta_x$ of the state-space $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ form an group denoted by $\mathrm{Aut}(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right])$. We will index the state $\Theta_x(g)$, input $\Theta_u(g)$, and output $\Theta_y(g)$ transformations by $g \in \mathrm{Aut}(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right])$.

*Proposition 2:* The set $\mathrm{Aut}(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right])$ of all symmetries $\Theta_y \in \mathbb{C}^{n_y \times n_y}$, $\Theta_u \in \mathbb{C}^{n_u \times n_u}$, and $\Theta_x \in \mathbb{C}^{n_x \times n_x}$ of the state-space $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is an infinite group.

*Proof:* See [17]. ■

Obviously the transfer function matrix $G(s) = C(sI - A)^{-1} B + D$ is symmetric (2) with respect to the symmetries $\Theta_y, \Theta_u \in \mathrm{Aut}(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right])$ of the state-space $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ since

$$\begin{aligned} \Theta_y G(s) &= \Theta_y C(sI - A)^{-1} B + \Theta_y D \\ &= C(sI - A)^{-1} B \Theta_u + D \Theta_u = G(s) \Theta_u \end{aligned}$$

Thus $\mathrm{Aut}(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]) \subseteq \mathrm{Aut}(G)$. The following theorem shows that the converse also holds.

*Theorem 2:* Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a minimal realization of the stable transfer function matrix $G(s)$. Then the state-space $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is symmetric with respect to the group $\mathrm{Aut}(G)$.

*Proof:* We need to show that for each $g \in \mathrm{Aut}(G)$ there exists a representation $\Theta_x(g) \in \mathbb{C}^{n_x \times n_x}$ that satisfies (3). Define

$$\Theta_x(g) = W_o^{-1} \int_0^{\infty} e^{A^\intercal \tau} C^\intercal \Theta_y(g) C e^{A \tau} d\tau \tag{4}$$

where the observability Grammian $W_o = \int_0^{\infty} e^{A^\intercal \tau} C^\intercal C e^{A \tau} d\tau$ is invertible since $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a minimal realization of $G(s)$.

Since $G(s) = \Theta_y(g) G(s) \Theta_u(g)^{-1}$ is symmetric (2), its impulse response is also symmetric

$$\Theta_y(g) D = D \Theta_u(g)$$
$$\Theta_y(g) C e^{At} B = C e^{At} B \Theta_u(g)$$

for all $t \in \mathbb{R}_+$ and $g \in \mathrm{Aut}(G)$. Therefore (4) is equivalent to

$$\begin{aligned}
\Theta_x(g) &= W_o^{-1}\int_0^\infty e^{A^\intercal \tau} C^\intercal \Theta_y(g) C e^{A\tau} d\tau W_c W_c^{-1} \\
&= W_o^{-1}\int_0^\infty \int_0^\infty e^{A^\intercal \tau} C^\intercal \Theta_y(g) C e^{A(\tau+\sigma)} B B^\intercal e^{A^\intercal \sigma} d\tau d\sigma W_c^{-1} \\
&= \int_0^\infty e^{A\sigma} B \Theta_u(g) B^\intercal e^{A^\intercal \sigma} d\sigma W_c^{-1}
\end{aligned} \tag{5}$$

where the controllability Grammian $W_c = \int_0^\infty e^{A\tau} B B^\intercal e^{A^\intercal \tau} d\tau$ is invertible since $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a minimal realization.

From (4) we have $\Theta_x B = B \Theta_u$ since

$$\begin{aligned}
\Theta_x(g) B &= W_o^{-1}\int_0^\infty e^{A^\intercal \tau} C^\intercal \Theta_y(g) C e^{A\tau} B d\tau \\
&= W_o^{-1}\int_0^\infty e^{A^\intercal \tau} C^\intercal C e^{A\tau} d\tau B \Theta_u(g) = B\Theta_u(g)
\end{aligned}$$

for all $g \in \mathrm{Aut}(G)$. Likewise from (5) we have

$$\begin{aligned}
C\Theta_x(g) &= \int_0^\infty C e^{A\tau} B \Theta_u(g) B^\intercal e^{A^\intercal \tau} d\tau W_c^{-1} \\
&= \Theta_y(g) C \int_0^\infty e^{A\tau} B B^\intercal e^{A^\intercal \tau} d\tau W_c^{-1} = \Theta_y(g) C
\end{aligned}$$

for all $g \in \mathrm{Aut}(G)$.

Since the controllability grammian $W_c$ is the unique positive definition solution to the Lyapunov equation $AW_c + W_c A^\intercal = -BB^\intercal$ we have

$$\begin{aligned}
&A\Theta_x(g) W_c - \Theta_x(g) A W_c \\
&= \int_0^\infty A e^{A\tau} B \Theta_u B^\intercal e^{A^\intercal \tau} d\tau - \int_0^\infty e^{A\tau} B \Theta_u B^\intercal e^{A^\intercal \tau} W_c^{-1} A W_c d\tau \\
&= \int_0^\infty \frac{d}{d\tau}\Big( e^{A\tau} B \Theta_u B^\intercal e^{A^\intercal \tau}\Big) d\tau + \int_0^\infty e^{A\tau} B \Theta_u B^\intercal e^{A^\intercal \tau} W_c^{-1} B B^\intercal d\tau \\
&= -B\Theta_u B^\intercal + \Theta_x B B^\intercal = 0
\end{aligned}$$

where we have already shown $\Theta_x(g) B = B\Theta_u(g)$. Thus we have $(A\Theta_x(g) - \Theta_x(g)A)W_c = 0$ which implies $A\Theta_x(g) = \Theta_x(g)A$ since $W_c \succ 0$.

Finally we show $\Theta_x$ is a representation of $\mathrm{Aut}(G)$. Note that for all $g, h \in \mathrm{Aut}(G)$ we have

$$\begin{aligned}
&\Theta_x(g)\Theta_x(h) \\
&= W_o^{-1}\int_0^\infty e^{A^\intercal \tau} C^\intercal \Theta_y(g) C e^{A\tau} d\tau \int_0^\infty e^{A\sigma} B \Theta_u(h) B^\intercal e^{A^\intercal \sigma} d\sigma W_c^{-1} \\
&= W_o^{-1}\int_0^\infty e^{A^\intercal \tau} C^\intercal \Theta_y(g)\Theta_y(h) C e^{A\tau} d\tau \int_0^\infty e^{A\sigma} B B^\intercal e^{A^\intercal \sigma} d\sigma W_c^{-1} \\
&= W_o^{-1}\int_0^\infty e^{A^\intercal \tau} C^\intercal \Theta_y(g)\Theta_y(h) C e^{A\tau} d\tau = \Theta_x(gh).
\end{aligned}$$

■

This theorem establishes an equivalence between input-output and state-space symmetries $\mathrm{Aut}(G) = \mathrm{Aut}(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right])$. Thus without ambiguity we can refer to *the* symmetry group $\mathrm{Aut}(G) = \mathrm{Aut}(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right])$ of a linear system.

The following corollary of Theorem 2 allows us to compute the state-space symmetry $\Theta_x(g)$ for $g \in \mathrm{Aut}(G)$ of a realization $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ from the input-output symmetries $\Theta_y(g)$ and $\Theta_u(g)$.

*Corollary 1:* Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a minimal realization of the stable transfer function matrix $G(s)$. Let $\Theta_y(g)$ and $\Theta_u(g)$ for $g \in \mathrm{Aut}(G)$ be input-output symmetries of the system. Then the state-space symmetry $\Theta_x(g)$ is given by

$$\Theta_x(g) = W_o^{-1}\hat{W}_o(g) = \hat{W}_c(g) W_c^{-1} \tag{6}$$

where $W_o$ and $W_c$ are the observability and controllability Grammians and $\hat{W}_o(g)$ and $\hat{W}_c(g)$ are the solutions to the Lyapunov equations

$$A^\intercal \hat{W}_o(g) + \hat{W}_o(g) A + C^\intercal \Theta_y(g) C = 0 \tag{7a}$$

$$A\hat{W}_c(g) + \hat{W}_c(g) A^\intercal + B\Theta_u(g) B^\intercal = 0. \tag{7b}$$

*Proof:* The solution $\hat{W}_o(g)$ of the Lyapunov equations (7a) is the integral $\hat{W}_o(g) = \int_o^\infty e^{A^\intercal \tau} C^\intercal \Theta_y(g) C e^{A\tau} d\tau$. Thus the state-space symmetry $\Theta_x(g) = W_o^{-1}\hat{W}_o(g)$ is an alternative expression for (4). Likewise $\Theta_x(g) = \hat{W}_c(g) W_c^{-1}$ is an alternative expression for (5). ■

This result is important since often the input-output symmetries $\Theta_y(g)$ and $\Theta_u(g)$ will have an intuitive physical interpretation but the state symmetries $\Theta_x(g)$ will not. For instance if the system model $G(s) = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ was experimentally identified, then inputs and outputs will have a physical meaning but the states will not. Therefore the state-space symmetries $\Theta_x(g)$ cannot be intuitively identified. Using Corollary 1 we can calculate the state-space symmetries for any realization of the system from the intuitive input-output symmetries.

A subgroup of symmetries $\mathcal{H} \subset \mathrm{Aut}(G)$ is called a permutation group if its representations $\Theta_x$, $\Theta_u$, and $\Theta_y$ are permutation matrices. There are many advantages to working with permutation matrices including numerical robustness and lower computational complexity. Additionally many algorithms in computational group theory are only applicable for permutation groups [18]. In many applications the input-output symmetries $\Theta_y$ and $\Theta_u$ are permutation matrices. However the nature of the state-space symmetries (6) depends on the realization $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of the transfer function. Applying a similarity transformation $T \in \mathbb{R}^{n_x \times n_x}$ to the system

$$\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \sim \left[\begin{smallmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{smallmatrix}\right]$$

creates a new representation $T\Theta_x(g)T^{-1}$ of the symmetry group $\mathrm{Aut}(G)$. The following proposition shows that the controller canonical and observer canonical forms have state-space symmetries $\Theta_x$ that are permutation matices.

*Proposition 3:* Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be the controller canonical or observer canonical realization of the stable system $G(s)$. Let $\mathcal{H} \subset \mathrm{Aut}(G)$ be a subgroup of $\mathrm{Aut}(G)$ such that $\Theta_u(g)$ and $\Theta_y(g)$ are permutation matrices for all $g \in \mathcal{H}$. Then the state-space symmetry $\Theta_x(g)$ is a permutation matrix for all $g \in \mathcal{H}$.

Unfortunately it is well known that the controller and observer canonical forms are numerically ill-conditioned for high-order systems [19]. It is unclear whether their exists a similarity transformation $T$ that produces both permutation symmetries $\Theta_x(g)$ and numerical well-conditioned realizations.

The following example demonstrates the concept of symmetry in linear systems.

*Example 1:* Consider the two-mass system shown in Figure 1. This system can be modeled by the transfer function matrix

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \underbrace{\frac{\begin{bmatrix} ms^2+2bs+2k & bs+k \\ bs+k & ms^2+2bs+2k \end{bmatrix}}{(ms^2+3bs+3k)(ms^2+bs+k)}}_{G(s)} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \tag{8}$$

where $U_1(s)$ and $U_2(s)$ are the Laplace transform of the forces $u_1(t)$ and $u_2(t)$ on the two blocks, and $Y_1(s)$ and $Y_2(s)$ are the Laplace transform of the positions $y_1(t)$ and $y_2(t)$ of the blocks.

The symmetry group $\mathrm{Aut}(G)$ of the transfer function matrix $G(s)$ is the set of all matrices $\Theta_u \in \mathbb{C}^{2\times 2}$ and $\Theta_y \in \mathbb{C}^{2\times 2}$ of the form $\Theta_u = \Theta_y = \left[\begin{smallmatrix} \alpha & \beta \\ \beta & \alpha \end{smallmatrix}\right]$ where $\alpha, \beta \in \mathbb{C}$ and $\alpha^2 - \beta^2 \neq 0$. In particular for $\alpha_1 = 1$ and $\beta_1 = 0$, and $\alpha_2 = 0$ and $\beta_2 = 1$ we obtain a permutation subgroup

$$\mathcal{H} = \Big\{ \underbrace{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]}_{g_1}, \underbrace{\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]}_{g_1} \Big\} \subset \mathrm{Aut}(G) \tag{9}$$

where $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]^2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. The symmetry $\Theta_y(g_1) = \Theta_u(g_1) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is the identity element. This symmetry is called trivial since it says

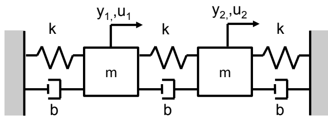


Fig. 1. Two-mass system. The control inputs $u_1$ and $u_2$ are the forces applied to the blocks and the outputs $y_1$ and $y_2$ are the positions of the blocks.

$G(s) = G(s)$. The reflective symmetry $\Theta_y(g_2) = \Theta_u(g_2) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ says that $G_{11}(s) = G_{22}(s)$ and $G_{12}(s) = G_{21}(s)$. Thus if we switch inputs $u_1(t)$ and $u_2(t)$, and switch outputs $y_1(t)$ and $y_2(t)$ the system input-output behavior does not change.

The two-mass system in Figure 1 can also be modeled using the state-space equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2k}{m} & -\frac{2b}{m} & \frac{k}{m} & \frac{b}{m} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m} & \frac{b}{m} & -\frac{2k}{m} & -\frac{2b}{m} \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m} \end{bmatrix}}_{B} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{10a}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \tag{10b}$$

where $x_1(t) = y_1(t)$ and $x_3(t) = y_2(t)$ are the positions of the blocks, and $x_2(t) = \dot{y}_1(t)$ and $x_4(t) = \dot{y}_2(t)$ are their velocities.

According to Theorem 2 every symmetry $g \in \mathrm{Aut}(G)$ of the transfer function matrix $G(s)$ corresponds to a symmetry $\Theta_x(g)$ of the state-space $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Note that (10) is in controller canonical form. By Proposition 3 the state-space symmetry $\Theta_x(g)$ should be a permutation matrix whenever $\Theta_u(g)$ and $\Theta_y(g)$ are permutation matrices. For the input-output symmetry $\Theta_y(g_2)$ and $\Theta_u(g_2)$ for $g_2 \in \mathcal{H} \subset \mathrm{Aut}(G)$ given by the permutation matrix $\Theta_u(g_2) = \Theta_y(g_2) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ we obtain the following state-space symmetry

$$\Theta_x(g_2) = W_o^{-1}\hat{W}_o(g) = \hat{W}_c(g)W_c^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

This symmetry permutes the positions and velocities of blocks 1 and 2. One can readily check that $\Theta_x$ satisfies the definition of state-space symmetry (3).

The state-space symmetry $\Theta_x$ depends on the realization $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ of the transfer function matrix (8). Consider the balanced realization of the transfer function matrix (8)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -0.3 & 1.3 & 0 & 0 \\ -1.3 & -0.6 & 0 & 0 \\ 0 & 0 & -0.7 & 2.1 \\ 0 & 0 & -2.1 & -2.3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} -0.4 & -0.4 \\ -0.4 & -0.4 \\ 0.3 & -0.3 \\ 0.3 & -0.3 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} -0.4 & 0.4 & 0.3 & -0.3 \\ -0.4 & 0.4 & -0.3 & 0.3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

where $m = 1$, $b = 1$, and $k = 2$. Under this realization, the state-space symmetry $\Theta_x$ is not a permutation matrix

$$\Theta_x(g_2) = W_o^{-1}\hat{W}_o(g) = \hat{W}_c(g)W_c^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Again one can readily check that $\Theta_x(g_2)$ satisfies (3). $\square$

## IV. Numerical Decomposition of Linear Systems

In this section we show that symmetric linear systems can be decomposed into decoupled subsystems. We provide an algorithm for performing this decomposition.

### A. *Structure of Linear Systems*

In this section we analyze the structural properties of symmetric linear systems. A similar analysis can be found in [2].

According to the definition (3) of state-space symmetry, the state-space matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ lies in the commutator-space

$$\mathcal{H}\left(\begin{bmatrix} \Theta_x & 0 \\ 0 & \Theta_y \end{bmatrix}, \begin{bmatrix} \Theta_x & 0 \\ 0 & \Theta_u \end{bmatrix}\right) = \left\{ M : \begin{bmatrix} \Theta_x & 0 \\ 0 & \Theta_y \end{bmatrix} M = M \begin{bmatrix} \Theta_x & 0 \\ 0 & \Theta_u \end{bmatrix}, \forall g \in \mathcal{H} \right\} \tag{11}$$

where $\Theta_{x,y,u} = \Theta_{x,y,u}(g)$, $M \in \mathbb{C}^{(n_x+n_y)\times(n_x+n_u)}$, and $\mathcal{H} \subset \mathrm{Aut}(G)$ is a finite-subgroup of $\mathrm{Aut}(G)$. According to the Structural Theorem 1 there exists orthogonal transformations

$$\Phi_x = \left[\Phi_x^{11}, \ldots, \Phi_x^{1r_1} | \ldots | \Phi_x^{p1}, \ldots, \Phi_x^{pr_p}\right] \in \mathbb{C}^{n_x \times n_x} \tag{12a}$$

$$\Phi_u = \left[\Phi_u^{11}, \ldots, \Phi_u^{1r_1} | \ldots | \Phi_u^{p1}, \ldots, \Phi_u^{pr_p}\right] \in \mathbb{C}^{n_u \times n_u} \tag{12b}$$

$$\Phi_y = \left[\Phi_y^{11}, \ldots, \Phi_y^{1r_1} | \ldots | \Phi_y^{p1}, \ldots, \Phi_y^{pr_p}\right] \in \mathbb{C}^{n_y \times n_y} \tag{12c}$$

that block-diagonalize the state-space

$$\begin{bmatrix} \Phi_x & 0 \\ 0 & \Phi_y \end{bmatrix}^* \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Phi_x & 0 \\ 0 & \Phi_u \end{bmatrix} \tag{13}$$

$$= \left[\begin{array}{ccc|ccc} I_{r_1} \otimes A_{11} & & & I_{r_1} \otimes B_{11} & & \\ & \ddots & & & \ddots & \\ & & I_{r_p} \otimes A_{pp} & & & I_{r_p} \otimes B_{pp} \\ \hline I_{r_1} \otimes C_{11} & & & I_{r_1} \otimes D_{11} & & \\ & \ddots & & & \ddots & \\ & & I_{r_p} \otimes C_{pp} & & & I_{r_p} \otimes D_{pp} \end{array}\right]$$

where $I_{r_i} \otimes A_{ii} = \bigoplus_{j=1}^{r_i} A_{ii}$ and

$$A_{ii} = (\Phi_x^{ij})^* A (\Phi_x^{ij}) \in \mathbb{R}^{n_x^i \times n_x^i} \tag{14a}$$

$$B_{ii} = (\Phi_x^{ij})^* B (\Phi_u^{ij}) \in \mathbb{R}^{n_x^i \times n_u^i} \tag{14b}$$

$$C_{ii} = (\Phi_y^{ij})^* C (\Phi_x^{ij}) \in \mathbb{R}^{n_y^i \times n_x^i} \tag{14c}$$

$$D_{ii} = (\Phi_y^{ij})^* D (\Phi_u^{ij}) \in \mathbb{R}^{n_y^i \times n_u^i} \tag{14d}$$

for $i = 1, \ldots, p$ and $j = 1, \ldots, r_i$. For short-hand we will write (13) as

$$\begin{bmatrix} \Phi_x & 0 \\ 0 & \Phi_y \end{bmatrix}^* \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Phi_x & 0 \\ 0 & \Phi_u \end{bmatrix} = \bigoplus_{i=1}^{p} I_{r_i} \otimes \begin{bmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{bmatrix}.$$

The transformations $\Phi_x$, $\Phi_y$, and $\Phi_u$ are partitioned as shown in equation (12). The submatrices

$$\Phi_x^i = [\Phi_x^{i1}, \ldots, \Phi_x^{ir_i}] \in \mathbb{C}^{r_i n_x^i \times r_i n_x^i}$$

$$\Phi_u^i = [\Phi_u^{i1}, \ldots, \Phi_u^{ir_i}] \in \mathbb{C}^{r_i n_u^i \times r_i n_u^i}$$

$$\Phi_y^i = [\Phi_y^{i1}, \ldots, \Phi_y^{ir_i}] \in \mathbb{C}^{r_i n_y^i \times r_i n_y^i}$$

of $\Phi_x$, $\Phi_u$, and $\Phi_y$ respectively, partition the state-space into $r_i$ repetitions of the block $\left[\begin{smallmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{smallmatrix}\right]$

$$\begin{bmatrix} \Phi_x^i & 0 \\ 0 & \Phi_y^i \end{bmatrix}^* \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Phi_x^i & 0 \\ 0 & \Phi_u^i \end{bmatrix} = I_{r_i} \otimes \begin{bmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{bmatrix}.$$

The submatrices $\Phi_x^{ij}$, $\Phi_u^{ij}$, and $\Phi_y^{ij}$ for $j = 1, \ldots, r_i$ produce the $r_i$ copies of the blocks $\left[\begin{smallmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{smallmatrix}\right]$. These blocks $\left[\begin{smallmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{smallmatrix}\right]$ are identical

$$\begin{bmatrix} \Phi_x^{ij} & 0 \\ 0 & \Phi_y^{ij} \end{bmatrix}^* \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Phi_x^{ij} & 0 \\ 0 & \Phi_u^{ij} \end{bmatrix} = \begin{bmatrix} \Phi_x^{ik} & 0 \\ 0 & \Phi_y^{ik} \end{bmatrix}^* \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \Phi_x^{ik} & 0 \\ 0 & \Phi_u^{ik} \end{bmatrix}$$

for all $j, k = 1, \ldots, r_i$. The $r_i$ repetitions of the identical blocks $\left[\begin{smallmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{smallmatrix}\right]$ reflects the intuition that symmetric systems have patterns that repeat.

The block-diagonalization $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] = \bigoplus_{i=1}^{p} I_{r_i} \otimes \left[\begin{smallmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{smallmatrix}\right]$ produces a corresponding block-diagonalization of the transfer function matrix $G(s) \in \mathcal{R}^{n_y \times n_u}$

$$\begin{aligned} \Phi_y^* G(s) \Phi_u &= \Phi_y^* C(sI - A)B\Phi_u + \Phi_y^* D \Phi_u \\ &= \textstyle\bigoplus_{i=1}^{p} \bigoplus_{j=1}^{r_i} G_{ii}(s) = \bigoplus_{i=1}^{p} I_{r_i} \otimes G_{ii}(s) \end{aligned}$$

where $G_{ii}(s) = C_{ii}(sI - A_{ii})^{-1}B_{ii} + D_{ii}$ is the transfer function matrix of the subsystem $\left[\begin{smallmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{smallmatrix}\right]$. In Section IV-B we will use this fact to block-diagonalize (13) the state-space $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ without explicitly computing the state-space transformation $\Phi_x \in \mathbb{C}^{n_x \times n_x}$.

### B. Decomposition using Minimal Realizations

There are several algorithms in the literature for computing the symmetry adapted basis $\Phi_x$, $\Phi_y$, and $\Phi_u$ [5], [6], [7]. However these algorithms are only applicable to permutation groups. In many applications the input and output symmetries $\Theta_u(g)$ and $\Theta_y(g)$ are permutation matrices. However, for numerically well-conditioned realizations, the state-space symmetries $\Theta_x(g)$ may not be permutation matrices. Thus it may not be possible to use these algorithms to compute $\Phi_x$. In this section we show how to implicitly compute the state-space transformation $\Phi_x \in \mathbb{C}^{n_x \times n_x}$ using minimal realizations of the transfer function matrix.

According to the definition of symmetry (2), for every $s \in \mathbb{C}$ the complex matrix $G(s) \in \mathbb{C}^{n_y \times n_u}$ lies in the commutator-space

$$\mathcal{H}\left(\Theta_y, \Theta_u\right) = \left\{ M \in \mathbb{C}^{n_y \times n_u} : \Theta_y M = M \Theta_u, \forall g \in \mathcal{H} \right\} \tag{15}$$

where $\Theta_{y,u} = \Theta_{y,u}(g)$, and $\mathcal{H} \subset \mathrm{Aut}(G)$ is a finite subgroup of $\mathrm{Aut}(G)$. The transfer function commutator $\mathcal{H}(\Theta_y, \Theta_u)$ is a subspace of the state-space commutator (11) in the sense that for each $G \in \mathcal{H}(\Theta_y, \Theta_u)$ we have

$$\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix} \in \mathcal{H}\left( \begin{bmatrix} \Theta_x & 0 \\ 0 & \Theta_y \end{bmatrix}, \begin{bmatrix} \Theta_x & 0 \\ 0 & \Theta_u \end{bmatrix} \right).$$

Thus the symmetry adapted basis (12) of the state-space commutator (11) also block-diagonalize the transfer function commutator (15). The set of matrices $\{G(s) : s \in \mathbb{C}\}$ decomposes

$$\Phi_y^* G(s) \Phi_u = \begin{bmatrix} I_{r_1} \otimes G_{11}(s) & & \\ & \ddots & \\ & & I_{r_p} \otimes G_{pp}(s) \end{bmatrix} = \bigoplus_{i=1}^{p} I_{r_i} \otimes G_{ii}(s)$$

where $G_{ii}(s) \in \mathbb{C}^{n_y^i \times n_u^i}$ for every $s \in \mathbb{C}$. Furthermore the sets of matrix blocks $G_{ii}(s) = (\Phi_y^{ij})^* G(s) (\Phi_u^{ij})$ for $s \in \mathbb{C}$ form transfer function matrices

$$G_{ii}(s) = \left[ \begin{array}{c|c} A & B(\Phi_u^{ij}) \\ \hline (\Phi_y^{ij})^* C & (\Phi_y^{ij})^* D (\Phi_u^{ij}) \end{array} \right] \in \mathcal{R}^{n_y^i \times n_u^i}. \tag{16}$$

The state-space (16) is potentially a non-minimal realization the transfer function blocks $G_{ii}(s)$. The following theorem shows that by realizing the transfer function blocks $G_{ii}(s)$ we recover the decoupled state-space blocks $\left[\begin{smallmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{smallmatrix}\right]$.

*Theorem 3:* Let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a minimal realization of the symmetric system $G(s)$. Let $\Phi_y$ and $\Phi_u$ be the symmetry adapted basis of the transfer function commutator (15). Then a minimal realization of transfer function matrix (16) is equivalent to the state-space block $\left[\begin{smallmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{smallmatrix}\right]$.

*Proof:* All minimal realizations of a system are equivalent. Therefore it suffices to prove that $\left[\begin{smallmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{smallmatrix}\right]$ is a minimal realization of (16).

By assumption $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a minimal realization of $G(s)$. Therefore (13) is a minimal realization of the system $\Phi_y^* G(s) \Phi_u$ since $\Phi_x \in \mathbb{C}^{n_x \times n_x}$ is a similarity transformation. Thus the realization (13) is observable and controllable. From the structure of (13), it can be shown that its controllability Grammian is of the form

$$W_c = \begin{bmatrix} I_{r_1} \otimes W_c^{11} & & \\ & \ddots & \\ & & I_{r_p} \otimes W_c^{pp} \end{bmatrix}.$$

Since the controllability Grammian $W_c$ is positive definite $W_c \succ 0$, each of the sub-Grammians $W_c^{ii}$ must be positive definite $W_c^{ii} \succ 0$. Thus the state-space block $\left[\begin{smallmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{smallmatrix}\right]$ is controllable. Likewise it can be shown that $\left[\begin{smallmatrix} A_{ii} & B_{ii} \\ C_{ii} & D_{ii} \end{smallmatrix}\right]$ is observable.

Now consider the orthogonal similarity transformation $\Phi_x$ applied to the transfer function block (16). This yields

$$\left[ \begin{array}{cccc|c} A_{11} & & & & B_{11} \\ & I_{r_1 - 1} \otimes A_{11} & & & 0 \\ & & \ddots & & \vdots \\ & & & I_{r_p} \otimes A_{pp} & 0 \\ \hline C_{11} & 0 & \cdots & 0 & D_{11} \end{array} \right]$$

where we have assumed without loss of generality that $i = 1$ and $j = 1$. Clearly the minimal realization of this system is a subsystem of $\left[\begin{smallmatrix} A_{11} & B_{11} \\ C_{11} & D_{11} \end{smallmatrix}\right]$. Since $\left[\begin{smallmatrix} A_{11} & B_{11} \\ C_{11} & D_{11} \end{smallmatrix}\right]$ is both controllable and observable it is a minimal realization of the transfer function block (16). ■

The following example demonstrates the decomposition of a linear system.

*Example 2:* Consider the two-mass system shown in Figure 1. A subgroup of symmetries for this system is the two-element set (9). The transfer function commutator $\mathcal{H}(\Theta_y, \Theta_u)$ is the set of matrices

$$\mathcal{H}(\Theta_y, \Theta_u) = \left\{ G \in \mathbb{C}^{2 \times 2} : \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] G = G \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \right\}$$

which contains the transfer function matrix (8). Using the algorithms from [5], [6], [7] we can find the following symmetry adapted basis $\Phi_y = [\Phi_y^{11} | \Phi_y^{21}]$ and $\Phi_u = [\Phi_u^{11} | \Phi_u^{21}]$ which block-diagonalize the commutator $\mathcal{H}(\Theta_y, \Theta_u)$

$$\Phi_y = \Phi_u = \tfrac{1}{\sqrt{2}} \left[ \begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array} \right].$$

For this example, the decoupled blocks $G_{ii}(s)$ do not repeat $r_1 = r_2 = 1$ and the blocks $G_{ii}(s) \in \mathbb{C}^{n_y^i \times n_u^i}$ have dimensions $n_y^1 = n_y^2 = n_u^1 = n_u^2 = 1$. The transformed input $\hat{u}_1(t) = \Phi_u^{11} u(t) = \big(u_1(t) + u_2(t)\big)/\sqrt{2}$ is the combined force applied to the blocks and the transformed output $\hat{y}_1(t) = \Phi_y^{11} y(t) = \big(y_1(t) + y_2(t)\big)/\sqrt{2}$ is the combined position of the blocks. The transformed input $\hat{u}_2(t) = \Phi_u^{21} u(t) = \big(u_1(t) - u_2(t)\big)/\sqrt{2}$ is differential force applied to the blocks and the transformed output $\hat{y}_2(t) = \Phi_y^{21} y(t) = \big(y_1(t) - y_2(t)\big)/\sqrt{2}$ differential position of the blocks.

Applying the symmetry adapted basis $\Phi_y$ and $\Phi_u$ to the transfer function matrix (8) produces the decoupled transfer function matrix

$$\Phi_y^* G(s) \Phi_u = \begin{bmatrix} \frac{1}{ms^2 + bs + k} & 0 \\ 0 & \frac{1}{ms^2 + 3bs + 3k} \end{bmatrix}.$$

The transfer function $G_{11}(s) \in \mathcal{R}^{1 \times 1}$ models the combined dynamics of the blocks and the transfer function $G_{22}(s) \in \mathcal{R}^{1 \times 1}$ models the differential dynamics of the blocks.

The symmetry adapted basis reduce the forth-order transfer function $G(s)$ into two second-order transfer functions $G_{11}(s)$ and $G_{22}(s)$. The reduction in order of $G_{11}(s) = \left[\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right] G(s) \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $G_{22}(s) = \left[\begin{smallmatrix} 1 & -1 \end{smallmatrix}\right] G(s) \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$ is due to pole-zero cancellations which separate the decoupled poles and zeros of the system. Although the poles and zeros are cancelled, they are not destroyed; poles and zeros cancelled in $G_{11}(s)$ appear in $G_{22}(s)$ and vice-versa.

These pole-zero cancellations are automatically calculated when compute a minimal realization of the transfer functions $G_{11}(s)$ and $G_{22}(s)$. Realizing the transfer functions $G_{11}(s)$ and $G_{22}(s)$ we obtain the following decoupled state-space model of the system

$$\frac{d}{dt} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & & \\ -\frac{k}{m} & -\frac{b}{m} & & \\ & & 0 & 1 \\ & & -\frac{3k}{m} & -\frac{3b}{m} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} + \begin{bmatrix} 0 & \\ \frac{1}{m} & \\ & 0 \\ & \frac{1}{m} \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}$$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & & \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix}$$

where $\hat{x}_1 = \hat{y}_1$ and $\hat{x}_3 = \hat{y}_2$ and $\hat{x}_2 = \frac{d\hat{y}_1}{dt}$ and $\hat{x}_4 = \frac{d\hat{y}_2}{dt}$. In general the states will not have physical meaning when realizing the transfer function matrices.

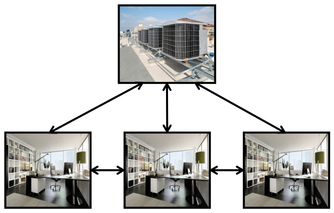

Fig. 2. Conceptual block-diagram of a Heating, Ventilation, and Air Conditioning System. HVAC system includes centralize components which interact with rooms. Rooms interact with each other.

## V. Application: HVAC

In this section we apply the results of this paper to the control of heating, ventilation, and air conditioning (HVAC) of buildings.

Building HVAC control systems are used to regulate room temperature in buildings. Figure 2 shows a conceptual diagram of an HVAC system. The HVAC system has centralize component which include compressors, condensers, and ventilation ports. There are components associated with each room which include electric heaters, air-conditioner evaporators, and ventilation dampers. The dynamics of the rooms and centralized components are strongly coupled.

The dynamics of an $N$ room HVAC system can be modeled by a transfer function matrix of the form

$$G(s) = \begin{bmatrix} G_{00}(s) & G_{01}(s) & G_{02}(s) & \dots & G_{0N}(s) \\ G_{10}(s) & G_{11}(s) & G_{12}(s) & \dots & G_{1N}(s) \\ G_{20}(s) & G_{12}(s) & G_{22}(s) & & G_{2N}(s) \\ \vdots & \vdots & & \ddots & \vdots \\ G_{N0}(s) & G_{1N}(s) & \dots & \dots & G_{NN}(s) \end{bmatrix} \tag{17}$$

where $G_{00}(s) \in \mathcal{R}^{n_y^0 \times n_u^0}$ and $G_{i0} \in \mathcal{R}^{n_y^i \times n_u^0}$ model the effects of the centralized inputs on the centralized outputs and $i$-th room outputs respectively, and $G_{0j} \in \mathcal{R}^{n_y^0 \times n_u^j}$ and $G_{ij} \in \mathcal{R}^{n_y^i \times n_u^j}$ for $j = 1, \dots, N$ model the effects the $j$-th room inputs on the central outputs and $i$-th room outputs respectively. The transfer function submatrices $G_{ij}(s) \in \mathcal{R}^{n_y^i \times n_u^j}$ for $i, j = 0, \dots, N$ are not necessarily square.

We assume that the rooms have similar dynamics, so that $G_{0i} = G_{0j}$, $G_{i0} = G_{j0}$, $G_{ii} = G_{jj}$, and $G_{ij} = G_{kl}$ for all $i \neq j, k \neq l \in \{1, \dots, N\}$. Under this assumption, a symmetry group for this system is the set of permutation matrices

$$\Theta_y(g) = \begin{bmatrix} I_{n_y^0} & 0 \\ 0 & \Pi(g) \otimes I_{n_y^i} \end{bmatrix} \text{ and } \Theta_u(g) = \begin{bmatrix} I_{n_u^0} & 0 \\ 0 & \Pi(g) \otimes I_{n_u^i} \end{bmatrix}$$

where $\Pi(g) \in \mathbb{R}^{N \times N}$ is any $N \times N$ permutation matrix. This symmetry group says that since the rooms are similar, we can permute them without changing the system dynamics. The transfer function commutator $\mathcal{H}(\Theta_y, \Theta_u)$ is the matrix set

$$\mathcal{H}(\Theta_y, \Theta_u) = \left\{ \begin{bmatrix} E & F & F & \dots & F \\ K & G & H & \dots & H \\ K & H & G & \dots & H \\ \vdots & \vdots & & \ddots & \vdots \\ K & H & \dots & \dots & G \end{bmatrix} : \begin{array}{l} E \in \mathbb{C}^{n_y^0 \times n_u^0}, F, K^* \in \mathbb{C}^{n_y^0 \times n_u^i}, \\ G, H \in \mathbb{C}^{n_y^i \times n_u^i} \end{array} \right\}.$$

Lets consider the case where $N = 4$. Then a set of symmetry adapted basis of this commutator-space are

$$\Phi_{y,u} = \frac{1}{2} \begin{bmatrix} I_{n_{y,u}^0} & 0 & 0 & 0 & 0 \\ 0 & I_{n_{y,u}^i} & I_{n_{y,u}^i} & I_{n_{y,u}^i} & -I_{n_{y,u}^i} \\ 0 & I_{n_{y,u}^i} & I_{n_{y,u}^i} & -I_{n_{y,u}^i} & I_{n_{y,u}^i} \\ 0 & I_{n_{y,u}^i} & -I_{n_{y,u}^i} & I_{n_{y,u}^i} & I_{n_{y,u}^i} \\ 0 & I_{n_{y,u}^i} & -I_{n_{y,u}^i} & -I_{n_{y,u}^i} & -I_{n_{y,u}^i} \end{bmatrix}$$

where $I_{n_y} \in \mathbb{R}^{n_y \times n_y}$ and $I_{n_u} \in \mathbb{R}^{n_u \times n_u}$ are identity matrices. The symmetry adapted basis $\Phi_y$ and $\Phi_u$ are similar for arbitrary $N$. Applying these symmetry adapted basis to the transfer function (17) produces the block-diagonal system

$$\Phi_y^* G(s) \Phi_u = \begin{bmatrix} G_{00} & \sqrt{N} G_{0i} & & & \\ \sqrt{N} G_{i0} & G_{ii}(s) + (N-1) G_{ij} & & & \\ & & G_{ii} - G_{ij} & & \\ & & & \ddots & \\ & & & & G_{ii} - G_{ij} \end{bmatrix}.$$

Our original $(n_y^0 + N n_y^i) \times (n_u^0 + N n_u^i)$ dimensional systems has been decomposed into one $(n_y^0 + n_y^i) \times (n_u^0 + n_u^i)$ dimensional system and $N-1$ systems of dimensions $n_y^i \times n_u^i$. Furthermore these $N-1$ systems $G_{ii}(s) - G_{ij}(s)$ are repeated. Exploiting this structure, we can design a centralized controller for the subsystem

$$(\Phi_y^{11})^* G(s) (\Phi_u^{11}) = \begin{bmatrix} G_{00} & \sqrt{N} G_{0i} \\ \sqrt{N} G_{i0} & G_{ii}(s) + (N-1) G_{ij} \end{bmatrix}$$

parameterized by the number of rooms $N$ and one controller for the repeated subsystem

$$(\Phi_y^{2k})^* G (\Phi_u^{2k}) = G_{ii}(s) - G_{ij}(s).$$

This controller can then be used for buildings with different numbers of rooms $N$ without the need to redesign the controller.

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