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A proportional integral extremum-seeking control approach for discrete-time nonlinear systems

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I. Introduction

Extremum-seeking control (ESC) has grown to become the leading approach to solve real-time optimization problems [1]. Following the seminal work of Krstic and coworkers ([2], [3], [4], [5], [6], [7]), this strikingly general and practically relevant control approach is equipped with an established and well understood control theoretical framework. The main drawback of ESC is the lack of transient performance guarantees. As highlighted in the proof of Krstic and Wang [2], the stability analysis relies on two components: an averaging analysis of the persistently perturbed ESC loop and a time-scale separation of ESC closed-loop dynamics between the fast transients of the system dynamics and the slow quasi steady-state extremum-seeking task. While the averaging analysis highlights the stability properties of ESC systems, the need for a slower time-scale for the optimization dynamics invariantly leads to a slow performance of the closed-loop ESC system. The objective of this study is to develop an ESC technique that minimizes the impact of time-scale separation on the transient performance of ESC systems for a class of discrete-time nonlinear dynamical systems.

The vast majority of existing results on ESC have focussed on continuous-time systems. Although discrete-time systems can be treated in an essentially similar fashion, the application of gradient descent in a discrete-time setting requires some care. A discrete-time version of the standard ESC loop was studied in [4] and [6] where convergence results similar to continuous time systems are obtained. A similar algorithm was also proposed in [8] for the tuning of PID controllers in unknown dynamical systems using ESC. Discrete-time ESC subject to stochastic perturbations is studied in [9]. The use of approximate parameterizations of the unknown cost function using quadratic functions was recently proposed in [10]. An alternative ESC-like approach was proposed in [11]. In this study, a trajectory based approach is used to analyze the properties of nonlinear optimization algorithms as dynamical systems. It is shown that properties of the nonlinear-optimization algorithms are suitable to assess the convergence of certain classes of ESC applied in a sampled-data approach. This approach was recently studied in the context of global sampling methods in [12] where trajectory based properties of nonlinear optimization methods are used to establish robust convergence. The main objectives with the trajectory based techniques is to analyze the properties of optimization algorithms assuming that they can converge to the true optimum using only the measurement of the objective function and possibly the constraints. In the context of ESC, one must either imply that the nonlinear optimization techniques do not rely on gradient information or, if they do, this gradient must be either measured or estimated. Some techniques such as [13] and [14] make use of sporadic gradient measurements in extremum seeking control. Other techniques [15] go as far as requiring the existence of multiple (nearly) identical systems to enable the estimation of gradient information.

This paper proposes the design of a fast ESC for discrete-time systems. The approach is based on a proportional-integral ESC (PIESC) design technique initially proposed in [16]. The approach extends the time-varying discrete-time ESC technique proposed in [17]. The PIESC technique proposed here is a combination of an integral action which corresponds to the standard ESC control task used to identify the steady-state optimum and a proportional control action designed to ensure that the measured cost function can be optimized instantaneously. Under suitable assumption on the dynamics of the system and the cost function, this action can be shown to minimize the cost over short times while reaching the optimum steady-state conditions.

The paper is organized as follows. A problem description of the ESC problem along with the key assumptions is given in Section II. The proposed proportional-integral ESC controller is described in Section III. Simulation examples are presented in Section IV followed by brief conclusions and proposed future work in Section V.

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II. Problem description

We consider a class of nonlinear systems of the form:

$$x_{k+1} = x_k + f(x_k) + g(x_k)u_k \quad (1)$$
$$y_k = h(x_k) \quad (2)$$

where $x_k \in \mathbb{R}^n$ is the vector of state variables at time k, u_k is the input variable at time k taking values in $\mathcal{U} \subset \mathbb{R}$ and $y_k \in \mathbb{R}$ is the objective function at step k, to be minimized. It is assumed that $f(x_k)$ and $g(x_k)$ are smooth vector valued functions and that $h(x_k)$ is a smooth function.

The objective is to steer the system to the equilibrium x^* and u^* that achieves the minimum value of $y(= h(x^*))$. The equilibrium (or steady-state) map is the n dimensional vector $x = \pi(u)$ that solves the following equation:

$$f(\pi(u)) + g(\pi(u))u = 0.$$

The corresponding equilibrium cost function is given by:

$$y = h(\pi(u)) = \ell(u) \quad (3)$$

At equilibrium, the problem is reduced to finding the minimizer u^* of $y = \ell(u^*)$. In the following, we let $\mathcal{D}(u)$ represent a neighbourhood of the equilibrium $x = \pi(u)$.

The following additional assumption concerning the steady-state cost function $\ell(u)$ is required.

Assumption 1: The nonlinear system is such that

$$\nabla_x h(\pi(u))g(\pi(u))(u - u^*) \geq \alpha_u \|u - u^*\|^2$$

for some positive constant α_u $\forall u \in \mathcal{U}$.

Some additional assumptions are required concerning the cost function $h(x)$.

Assumption 2: The cost $h(x)$ is such that

1) $\frac{\partial h(x^*)}{\partial x} = 0$
2) $\frac{\partial^2 h(x)}{\partial x \partial x^T} > \beta I, \ \forall x \in \mathbb{R}^n$

where β is a strictly positive constant.

It is assumed that the cost function dynamics has a relative degree of one. The cost function dynamics are expressed as follows. We let $\alpha(x_k) = x_k + f(x_k) + g(x_k)\hat{u}_k$. The rate of change of the cost function $y_k = h(x_{k+1})$ is given by:

$$h(x_{k+1}) - h(x_k) = h(x_k + f(x_k) + g(x_k)u_k) - h(\alpha(x_k)) + h(\alpha(x_k)) - h(x_k).$$

The first two terms can be rewritten using the second order Taylor formula as:

$$h(x_k + f(x_k) + g(x_k)u_k) - h(\alpha(x_k)) = \nabla h(\alpha(x_k))g(x_k)(u_k - \hat{u}_k) + \frac{1}{2}(u_k - \hat{u}_k)^\top g(x_k)^\top \nabla^2 h(\tilde{y}_k)g(x_k)(u_k - \hat{u}_k) \quad (4)$$

where $y_k = \alpha(x_k) + \theta g(x_k)(u_k - \hat{u}_k)$ for $\theta \in (0, 1)$. We rewrite (4) as follows:

$$h(x_k + f(x_k) + g(x_k)u_k) - h(\alpha(x_k)) = \quad (5)$$
$$\Psi_{1,k}(x_k, u_k, \hat{u}_k)(u_k - \hat{u}_k) \quad (6)$$

where

$$\Psi_{1,k}(x_k, u_k, \hat{u}_k) = (\nabla h(\alpha(x_k)))g(x_k) + \frac{1}{2}(u_k - \hat{u}_k)^\top g(x_k)^\top \nabla^2 h(\tilde{y}_k)g(x_k).$$

We also define the following

$$\Psi_{0,k}(x_k, \hat{u}_k) = h(\alpha(x_k, \hat{u}_k)) - h(x_k).$$

and write the cost dynamics as:

$$y_{k+1} - y_k = \Psi_{0,k}(x_k, \hat{u}_k) + \Psi_{1,k}(x_k, u_k, \hat{u}_k)(u_k - \hat{u}_k).$$

By the relative order one assumption on $h(x)$, the system's dynamics can be decomposed and written as:

$$\xi_{k+1} = \xi_k + \psi(\xi_k, y_k) \quad (7)$$
$$y_{k+1} = y_k + \Psi_{0,k}(x_k, \hat{u}_k) + \Psi_{1,k}(x_k, u_k, \hat{u}_k)(u_k - \hat{u}_k) \quad (8)$$

where $\xi_k \in \mathbb{R}^{n-1}$ and $\psi(\xi_k, y_k)$ is a smooth vector valued function.

Assumption 3: There exists a positive definite function $W(\xi)$ that satisfies the following inequalities:

$$\beta_1 \|x_k - \pi(\hat{u})\|^2 \leq W(\xi) + h(x) \leq \beta_2 \|x_k - \pi(\hat{u})\|^2$$

with positive constants β_1 and β_2, and:

$$W(\xi_{k+1}) + h(\alpha(x_k)) - W(\xi_k) - h(x_k) \leq -\alpha_e \|x_k - \pi(\hat{u})\|^2$$

with positive constant α_e, $\forall x_k \in \mathcal{D}(\hat{u})$ and $\forall \hat{u} \in \mathcal{U}$.
Assumption 3 states that $W + h$ is non-increasing along the vector field $f(x) + g(x)u$ over some neighbourhood of the steady-state manifold $x = \pi(u)$ at a fixed value of the input $\hat{u}$.

III. Proportional-Integral Perturbation Discrete-time ESC

In this section, we present the proposed ESC controller.

Recall that the cost function dynamics can be parameterized as follows:

$$y_{k+1} = y_k + \theta_{0,k} + \theta_{1,k}(u_k - \hat{u}_k)$$

where the time-varying parameters $\theta_{0,k}$ and $\theta_{1,k}$ are identified with $\theta_{0,k} = \Psi_{0,k}$ and $\theta_{1,k} = \Psi_{1,k}$.

Since the parameters $\theta_{0,k}$ and $\theta_{1,k}$ are unknown, they must be estimated. Let $\hat{\theta}_{0,k}$ and $\hat{\theta}_{1,k}$ denote the estimates of $\theta_{0,k}$ and $\theta_{1,k}$, respectively. The proposed proportional-integral extremum-seeking controller is given by:

$$u_k = -k_g \hat{\theta}_{1,k} + \hat{u}_k \quad (9)$$
$$\hat{u}_{k+1} = \hat{u}_k - \frac{1}{\tau_I}\hat{\theta}_{1,k}.$$

where k_g and τ_I are positive constants to be assigned.

A. Time-varying parameter estimation approach

This section describes a scheme that allows the accurate estimation of the parameters $\theta_{0,k}$ and $\theta_{1,k}$. Note that the estimation $\theta_{0,k}$ is necessary to ensure that the estimates of $\theta_{1,k}$ are not biased.e

Consider the following state predictor

$$\begin{aligned}\hat{y}_{k+1} &= \hat{y}_k + \hat{\theta}_{0,k} + \hat{\theta}_{1,k}(u_k - \hat{u}_k) \\ &+ K_k e_k - \omega_{k+1}(\hat{\theta}_k - \hat{\theta}_{k+1}) \end{aligned} \quad (10)$$

where $\hat{\theta}_k = [\hat{\theta}_{0,k}, \hat{\theta}_{1,k}^T]^T$ is the vector of parameter estimates at time step k given by any update law, K_k is a correction factor at time step k, $e_k = x_k - \hat{x}_k$ is the state estimation error at time step k. We $\phi_k = [1, (u_k - \hat{u}_k)^T]^T$. The variable ω_k is the following output filter at time step k

$$\omega_{k+1} = \omega_k + \phi_k - K_k\omega_k, \qquad \omega_0 = 0 \quad (11)$$

Using the state predictor defined in (10) and the output filter defined in (11), the prediction error $e_k = x_k - \hat{x}_k$ is given by

$$\begin{aligned} e_{k+1} &= e_k + G(x_k, u_k)\tilde{\theta}_{k+1} - K_k e_k \\ &+ \omega_{k+1}(\hat{\theta}_k - \hat{\theta}_{k+1}) + \omega_{k+1}(\theta_{k+1} - \theta_k) \\ e_0 &= x_0 - \hat{x}_0. \end{aligned} \quad (12)$$

An auxiliary variable η_k is introduced which is defined as $\eta_k = e_k - \omega_k^T\tilde{\theta}_k$. Its dynamics are described as follows

$$\begin{aligned} \eta_{k+1} &= e_{k+1} - \omega_{k+1}\tilde{\theta}_{k+1} \\ \eta_0 &= e_0. \end{aligned} \quad (13)$$

Since ϑ_k is unknown, it is necessary to use an estimate, $\hat{\eta}$, of η . The estimate is generated by the recursion:

$$\hat{\eta}_{k+1} = \hat{\eta}_k - K_k\hat{\eta}_k \quad (14)$$

The resulting dynamics of the η estimation error are:

$$\tilde{\eta}_{k+1} = \tilde{\eta}_k - K_k\tilde{\eta}_k + \omega_{k+1}^T(\theta_{k+1} - \theta_k) \quad (15)$$

Let the identifier matrix Σ_k be defined as

$$\Sigma_{k+1} = \alpha\Sigma_k + \omega_k^T\omega_k, \quad \Sigma_0 = \alpha I \succ 0 \quad (16)$$

with an inverse generated by the recursion

$$\begin{aligned}\Sigma_{k+1}^{-1} =& \Sigma_k^{-1} + \left(\frac{1}{\alpha} - 1\right)\Sigma_k^{-1} \\ &- \frac{1}{\alpha^2}\Sigma_k^{-1}\omega_k(1 + \frac{1}{\alpha}\omega_k^T\Sigma_k^{-1}\omega_k)^{-1}\omega_k^T\Sigma_k^{-1} \end{aligned} \quad (17)$$

Using equations (10), (11), and (14), the parameter update law is

$$\hat{\theta}_{k+1} = \hat{\theta}_k + \Sigma_k^{-1}\omega_k^T\left(I + w_k\Sigma_k^{-1}w_k^T\right)^{-1}(e_k - \hat{\eta}_k) \quad (18)$$

To ensure that the parameter estimates remain within the constraint set Θ_k, we propose to use a projection operator of the form:

$$\bar{\hat{\theta}}_{k+1} = \text{Proj}\{\hat{\theta}_k + \Sigma_k^{-1}\omega_k^T\left(I + w_k\Sigma_k^{-1}w_k^T\right)^{-1}(e_k - \hat{\eta}_k), \Theta_k\} \quad (19)$$

The operator Proj represents an orthogonal projection onto the surface of the uncertainty set applied to the parameter estimate.The parameter uncertainty set is defined by the ball function $B(\hat{\theta}_c, z_{\hat{\theta}c})$, where $\hat{\theta}_c$ and $z_{\hat{\theta}c}$ are the parameter estimate and set radius found at the latest set update.

Following [18], the projection operator is designed such that

- $\bar{\hat{\theta}}_{k+1} \in \Theta_0$
- $\bar{\tilde{\theta}}_{k+1}^T\Sigma_{k+1}\bar{\tilde{\theta}}_{k+1} \leq \tilde{\theta}_{k+1}^T\Sigma_{k+1}\tilde{\theta}_{k+1}$

One possible algorithm for the projection algorithm is as follows. Define the upper bound for $\|\theta\|$ $(= L_1)$. Let $R =$Chol(Σ_{k+1}) denote the Cholesky factor of Σ_{k+1}. Then we perform the following:

Algorithm 1: If $\|\hat{\theta}_{k+1}\| \geq L_1$ then

- Let $\delta = \frac{L_1\hat{\theta}_{k+1}}{\|\hat{\theta}_{k+1}\|}$,
- Let $z_\rho = \sqrt{\delta^T\Sigma_{k+1}\delta}$,
- With $\rho = R\hat{\theta}_{k+1}$ define $\bar{\rho} = \frac{\rho z_\rho}{\|\rho\|}$,
- Let $\bar{\hat{\theta}}_{k+1} = R^{-1}\bar{\rho}$.

Otherwise,

- Let $\bar{\hat{\theta}}_{k+1} = \hat{\theta}_{k+1}$.

It is assumed that the trajectories of the system are such that the following condition is met.

Assumption 4: [18] There exists constants $\beta_T > 0$ and $T > 0$ such that

$$\frac{1}{T}\sum_{i=k}^{k+T-1}\omega_i\omega_i^T > \beta_T I, \ \forall k > T. \quad (20)$$

This requirement is a standard persistency of excitation condition that can be found in most references on adaptive control and adaptive estimation. The reader is referred to [18] for more details.

B. Main result

In this section, we present the main result of this study.

Theorem 1: Consider the nonlinear discrete-time system (1) with cost function (2), the extremum seeking controller (9) and parameter estimation scheme (10), (11), (14), (16) and (19). Let Assumptions 1-4 be fulfilled. Then there exists positive constants α, K, k_g and τ_I such that for every $\tau_I \geq \tau_I^*$, the states x_k and input u_k of the closed-loop system enter a neighbourhood of the unknown optimum (x^*, u^*).

Proof: Let $\tilde{u}_k = u_k - u^*$ and consider the Lyapunov function:

$$W_k = \tilde{\theta}_k^T\Sigma_k\tilde{\theta}_k.$$

Consider the following:

$$\begin{aligned} W_{k+1} - W_k &= \bar{\tilde{\theta}}_{k+1}^T\Sigma_{k+1}\bar{\tilde{\theta}}_{k+1} - \tilde{\theta}_k^T\Sigma_k\tilde{\theta}_k \\ &\leq \tilde{\theta}_{k+1}^T\Sigma_{k+1}\tilde{\theta}_{k+1} - \tilde{\theta}_k^T\Sigma_k\tilde{\theta}_k. \end{aligned} \quad (21)$$

where the final inequality arises as a result of the properties of the projection algorithm.

Let $Q_k = \left(1 + \frac{1}{\alpha}\omega_k^T\Sigma_k^{-1}\omega_k\right)^{-1}$. The parameter estimation error dynamics is given by:

$$\begin{aligned}\tilde{\theta}_{k+1} &= \tilde{\theta}_k + (\theta_{k+1} - \theta_k) - \frac{1}{\alpha}\Sigma_k^{-1}\omega_k Q_k(e_k - \hat{\eta}_k). \\ &= \tilde{\theta}_k + (\theta_{k+1} - \theta_k) - \frac{1}{\alpha}\Sigma_k^{-1}\omega_k Q_k\omega_k^T\tilde{\theta}_k \\ &- \frac{1}{\alpha}\Sigma_k^{-1}\omega_k Q_k\tilde{\eta}_k \end{aligned}$$

Note that by construction one can write the parameter estimation error dynamics as follows:

$$\tilde{\theta}_{k+1} = (\theta_{k+1} - \theta_k) + \alpha \Sigma_{k+1}^{-1} \Sigma_k \tilde{\theta}_k - \frac{1}{\alpha} \Sigma_k^{-1} \omega_k Q_k \tilde{\eta}_k \quad (22)$$

Upon successive substitution of $\tilde{\theta}_k$, one obtains the following by induction:

$$\begin{aligned} \tilde{\theta}_{k+1} =& \Sigma_{k+1}^{-1} \alpha^{k+1} \Sigma_0 \tilde{\theta}_0 + \Sigma_{k+1}^{-1} \sum_{i=1}^{k} \alpha^{k-i+1} \Sigma_i (\theta_{i+1} - \theta_i) \\ & - (1-K) \Sigma_{k+1}^{-1} \sum_{i=1}^{k} \alpha^{k-i-1} \Sigma_{i+1} \Sigma_i^{-1} \omega_i Q_i \tilde{\eta}_{i-1} \end{aligned}$$

The matrix Σ_{k+1} can be bounded as follows. The recursion for Σ_k can be rewritten as:

$$\Sigma_{k+1} = \alpha^{k+1} \Sigma_0 + \sum_{i=0}^{k} \alpha^{k-i} \omega_i \omega_i^T.$$

Then one can write:

$$\begin{aligned} \Sigma_{k+1} &\leq \alpha^{k+1} \Sigma_0 + \sum_{i=0}^{k} \alpha^{k-i} \sum_{j=1}^{T} \omega_{i+j} \omega_{i+j}^T \\ &\leq \alpha^{k+1} \Sigma_0 + \sum_{i=0}^{k} \alpha^{k-i} T \beta I \leq \alpha^{k+1} \Sigma_0 + \frac{1-\alpha^{k+1}}{1-\alpha} T \beta I \end{aligned}$$

Similarly, one can provide a lower bound for Σ_{k+1}. Consider the quantity:

$$T \Sigma_{k+1} = T \alpha^{k+1} \Sigma_0 + T \sum_{i=0}^{k} \alpha^{k-i} \omega_i \omega_i^T$$

Using a simple rearrangement of the summation term one obtains:

$$\begin{aligned} T \Sigma_{k+1} \geq & T \alpha^{k+1} \Sigma_0 + \sum_{i=T}^{k} \alpha^{k-i} \omega_i \omega_i^T + \sum_{i=T-1}^{k-1} \alpha^{k-i} \omega_i \omega_i^T + \\ & \ldots + \sum_{i=0}^{k-T} \alpha^{k-i} \omega_i \omega_i^T \end{aligned}$$

which leads to

$$T \Sigma_{k+1} \geq T \alpha^{k+1} \Sigma_0 + \sum_{i=0}^{k-T} \alpha^{k-i} \sum_{j=0}^{T-1} \alpha^{-j} \omega_{i+j} \omega_{i+j}^T$$

or,

$$T \Sigma_{k+1} \geq T \alpha^{k+1} \Sigma_0 + \sum_{i=0}^{k-T} \alpha^{k-i} \sum_{j=0}^{T-1} \omega_{i+j} \omega_{i+j}^T.$$

Invoking assumption 4 and rearranging, we can finally write:

$$T \Sigma_{k+1} \geq T \alpha^{k+1} \Sigma_0 + \frac{\alpha^T}{1-\alpha} T \beta_T I \geq \frac{\alpha^T}{1-\alpha} T \beta_T I.$$

Assuming that $\Sigma_0 = \alpha_0 I$, one gets the following bounds:

$$\frac{\alpha^T}{1-\alpha} \beta_T I \leq \Sigma_{k+1} \leq \alpha_0 I + \frac{1}{1-\alpha} T \beta I. \quad (23)$$

or,

$$\frac{1-\alpha}{\alpha_0 + T\beta} \leq \Sigma_{k+1}^{-1} \leq \frac{1-\alpha}{\beta_T \alpha^T} I. \quad (24)$$

By the dynamics of $\tilde{\eta}_k$, it is easy to show that:

$$\tilde{\eta}_{k+1} = \sum_{1}^{k} (1-K)^{k-i+1} \tilde{\eta}_0 + \sum_{i=1}^{k} (1-K)^{k-i} \omega_{i+1}^T (\theta_{i+1} - \theta_i)$$

As a result, one obtains the upper bound:

$$\begin{aligned} \|\tilde{\eta}_{k+1}\| \leq & \sum_{i=1}^{k} (1-K)^{k-i+1} \|\tilde{\eta}_0\| \\ & + \sum_{i=1}^{k} (1-K)^{k-i} \sqrt{\beta} \|(\theta_{i+1} - \theta_i)\| \end{aligned}$$

The parameter estimation error $\|\tilde{\theta}_{k+1}\|$ is such that:

$$\begin{aligned} & \|\tilde{\theta}_{k+1}\| \leq \frac{1-\alpha}{\beta_T \alpha^T} \alpha^{k+1} \alpha_0 \|\tilde{\theta}_0\| \\ & + \frac{1-\alpha}{\beta_T \alpha^T} \left(\sum_{i=1}^{k} \alpha^{k-i+1} \alpha_0 \|(\theta_{i+1} - \theta_i)\| \right. \\ & \left. + \sum_{i=1}^{k} \alpha^{k-i+1} \frac{1}{1-\alpha} T \beta \|(\theta_{i+1} - \theta_i)\| \right) \\ & + \left(\frac{1-\alpha}{\beta_T \alpha^T} \right)^2 \left(\sum_{i=1}^{k} \alpha^{k-i+1} (1-K)^{k-i} \alpha_0 \beta \|(\theta_{i+1} - \theta_i)\| \right. \\ & \left. + \sum_{i=1}^{k} \alpha^{k-i+1} (1-K)^{k-i} \frac{1}{1-\alpha} T \beta^2 \|(\theta_{i+1} - \theta_i)\| \right) \\ & + \left(\frac{1-\alpha}{\beta_T \alpha^T} \right)^2 \left(\sum_{i=1}^{k} \alpha^{k-i+1} (1-K)^{k-i+1} \alpha_0 \|\tilde{\eta}_0\| \right. \\ & \left. + \sum_{i=1}^{k} \alpha^{k-i+1} (1-K)^{k-i+1} \frac{1}{1-\alpha} T \beta \|\tilde{\eta}_0\| \right). \end{aligned}$$

By smoothness of $\Psi_{0,k}$ and $\Psi_{1,k}$, it follows that, $\forall x_k \in \mathcal{D}(u)$ and $\forall u \in \mathcal{U}$, the inequality:

$$\|\theta_{i+1} - \theta_i\| \leq \|\Psi_{0,i+1} - \Psi_{0,i}\| + \|\Psi_{1,i+1} - \Psi_{1,i}\|$$

can be written as:

$$\begin{aligned} \|\theta_{i+1} - \theta_i\| \leq & L_{\Psi_1} \|x_{k+1} - x_k\| + L_{\Psi_2} \|\hat{u}_{k+1} - \hat{u}_k\| \\ & + L_{\Psi_3} \|(u_{k+1} - \hat{u}_{k+1}) - (u_k - \hat{u}_k)\| \end{aligned}$$

where L_{Ψ_i}, $i = 1, 2, 3$, are Lipschitz constants. Upon substitution of the process dynamics and the extremum seeking controller, we obtain:

$$\begin{aligned} & \|\theta_{i+1} - \theta_i\| \leq L_{\Psi_1} \|f(x_k) + g(x_k)(-k_g \hat{\theta}_{1,k} + \hat{u}_k + d_k)\| \\ & + \frac{L_{\Psi_2}}{\tau_I} \|\hat{\theta}_{1,k}\| + k_g L_{\Psi_3} \|\hat{\theta}_{k+1} - \hat{\theta}_k\| \end{aligned}$$

This last inequality reduces to:

$$\begin{aligned} & \|\theta_{i+1} - \theta_i\| \leq L_{\Psi_1} L_F \|x_k - \pi(\hat{u}_k)\| \\ & + k_g L_{\Psi_1} L_G \|x_k - \pi(\hat{u}_k)\| \|\hat{\theta}_{1,k}\| \\ & + L_{\Psi_1} L_G \|x_k - \pi(\hat{u}_k)\| \|d_k\| + k_g L_{\Psi_1} \|g(\pi(\hat{u}_k))\| \|\hat{\theta}_{1,k}\| \\ & + L_{\Psi_1} \|g(\pi(\hat{u}_k))\| \|d_k\| + \frac{L_{\Psi_2}}{\tau_I} \|\hat{\theta}_{1,k}\| + k_g L_{\Psi_3} \|\hat{\theta}_{k+1} - \hat{\theta}_k\| \end{aligned}$$

where L_F and L_G are Lipschitz constants for the vector fields $f(x_k)$ and $g(x_k)$. Finally, we obtain inequality:

$$\|\theta_{i+1}-\theta_i\| \leq (L_{\Psi_1}L_F + k_g L_{\Psi_1}L_G + DL_{\Psi_1}L_G)\|x_k - \hat{u}_k\| + k_g L_{\Psi_1}GL_1 + DL_{\Psi_1}G + \frac{L_1 L_{\Psi_2}}{\tau_I} + 2k_g L_{\Psi_3}L_1$$

which we write as:

$$\|\theta_{i+1}-\theta_i\| \leq b_1(k_g, D)\|x_k - \pi(\hat{u}_k)\| + b_0(k_g, \frac{1}{\tau_I}, D).$$

Without loss of generality, we also assume that $\|\tilde{\eta}_0\| = 0$.

Then one can write:

$$\begin{aligned}\|\tilde{\theta}_{k+1}\| \leq & \frac{1-\alpha}{\beta_T}\alpha^{k-T+1}\alpha_0\|\tilde{\theta}_0\| \\ & + \Upsilon(T,\alpha,K)b_1(k_g,D)\|x_k - \pi(\hat{u}_k)\| \\ & + \Upsilon(T,\alpha,K)b_0(k_g,\frac{1}{\tau_I},D) = c_1 + c_2\|x_k - \pi(\hat{u}_k)\|\end{aligned}$$

where

$$\begin{aligned}\Upsilon(T,\alpha,K) = & \frac{1-\alpha^{k+1}}{\beta_T\alpha^T}\alpha_0 + \frac{1-\alpha^{k+1}}{\beta_T\alpha^T(1-\alpha)}T\beta \\ & + \frac{(1-\alpha)(1-\alpha^{k+1}(1-K)^{k+1})}{(1-K)(\beta_T\alpha^T)^2}\alpha_0\beta \\ & + \frac{1-\alpha^{k+1}(1-K)^{k+1}}{(1-K)(\beta_T\alpha^T)^2}T\beta^2.\end{aligned}$$

We thus see that the parameter estimation error will tend to a neighbourhood of the origin. The size of this neighbourhood depends primarily on the constant T associated with the persistency of excitation condition. Next we pose the following Lyapunov function candidate: $\mathcal{V} = W + h + \frac{1}{2}\tilde{u}^T\tilde{u}$. The recursion of $\mathcal{V}$ yields:

$$\begin{aligned}\mathcal{V}_{k+1} - \mathcal{V}_k = & W_{k+1} - W_k + \Psi_{0,k} + \Psi_{1,k}(u_k - \hat{u}_k) \\ & + \frac{1}{2}\tilde{u}_{k+1}^T\tilde{u}_{k+1} - \frac{1}{2}\tilde{u}_k^\top\tilde{u}_k.\end{aligned}$$

Substitution of the ESC yields:

$$\begin{aligned}\mathcal{V}_{k+1} - \mathcal{V}_k = & W_{k+1} - W_k + \Psi_{0,k} - k_g\Psi_{1,k}\hat{\theta}_{1,k} + \Psi_{1,k}d_k \\ & + \frac{1}{2}\left(\tilde{u}_k + \frac{1}{\tau_I}\hat{\theta}_{1,k}\right)^T\left(\tilde{u}_k + \frac{1}{\tau_I}\hat{\theta}_{1,k}\right) - \frac{1}{2}\tilde{u}_k^\top\tilde{u}_k.\end{aligned}$$

Replacing $\hat{\theta}_{1,k} = \Psi_{1,k} - \tilde{\theta}_{1,k}$ and using assumptions 1 and 3, one obtains:

$$\begin{aligned}\mathcal{V}_{k+1} - \mathcal{V}_k \leq & -\alpha_e\|x - \pi(\hat{u}_k)\|^2 - \left(k_g - \frac{1}{2\tau_I^2}\right)\|\Psi_{1,k}\|^2 \\ & + \left|\left(k_g - \frac{1}{\tau_I^2}\right)\right|\|\Psi_{1,k}\|\|\tilde{\theta}_{1,k}\| + \|\Psi_{1,k}\|\|d_k\| \\ & - \frac{\alpha_u}{\tau_I}\|\tilde{u}_k\|^2 + \frac{L_H}{\tau_I}\|x - \pi(\hat{u}_k)\|\|\tilde{u}_k\| + \frac{1}{\tau_I}\|\tilde{u}_k\|\|\tilde{\theta}_{1,k}\| \\ & + \frac{1}{2\tau_I^2}\|\tilde{\theta}_{1,k}\|^2\end{aligned}$$

where L_H is the Lipschizt constant associated with

$$\|\Psi_{1,k} - \nabla h(\hat{u}_k)g(\pi(\hat{u}_k))\| \leq L_H\|x - \pi(\hat{u}_k)\|.$$

Substituting for the upper bound of $\|\tilde{\theta}_k\|$, rearranging and letting $k_g = \frac{1}{\tau_I^2}$, one obtains:

$$\begin{aligned}\mathcal{V}_{k+1} - \mathcal{V}_k \leq & -\begin{bmatrix}\|x - \pi(\hat{u}_k)\| & \|\tilde{u}_k\| & \|\Psi_{1,k}\|\end{bmatrix} \\ & \times\begin{bmatrix}\alpha_e - \frac{c_2L_H}{\tau_I} - \frac{c_2^2}{\tau_I^2} & -\frac{L_H}{2\tau_I} & 0 \\ -\frac{L_H}{2\tau_I} & \frac{\alpha_u}{\tau_I} & 0 \\ 0 & 0 & \left(\frac{1}{\tau_I^2}\right) - \frac{c_2L_H}{\tau_I} - \frac{c_2^2}{\tau_I^2}\end{bmatrix} \\ & \times\begin{bmatrix}\|x - \pi(\hat{u}_k)\| \\ \|\tilde{u}_k\| \\ \|\Psi_{1,k}\|\end{bmatrix} \\ & + \frac{c_1L_H}{\tau_I}\|x - \pi(\hat{u}_k)\| + \frac{c_1}{\tau_I}\|\tilde{u}_k\| + (D)\|\Psi_{1,k}\| + \frac{c_1^2}{\tau_I^2}\end{aligned}$$

It is to see that there exists a τ_I^* such that $\forall\tau_I > \tau_I^*$, with $k_g = \frac{1}{\tau_I^2}$, the last inequality can be written as:

$$\begin{aligned}\mathcal{V}_{k+1} - \mathcal{V}_k \leq & -\lambda_1\|x - \pi(\hat{u}_k)\|^2 - \lambda_1\|\tilde{u}_k\|^2 \\ & - \lambda_1\|\Psi_{1,k}\|^2 + \frac{c_1L_H}{\tau_I}\|x - \pi(\hat{u}_k)\| \\ & + \frac{c_1}{\tau_I}\|\tilde{u}_k\| + D\|\Psi_{1,k}\| + \frac{c_1^2}{\tau_I^2}\end{aligned}$$

for a positive constant $\lambda_1 > 0$ taken as the minimum eigenvalue of the matrix:

$$\begin{bmatrix}\alpha_e - \frac{c_2L_H}{\tau_I} - \frac{c_2^2}{\tau_I^2} & -\frac{L_H}{2\tau_I} & 0 \\ -\frac{L_H}{2\tau_I} & \frac{\alpha_u}{\tau_I} & 0 \\ 0 & 0 & \left(\frac{1}{\tau_I^2}\right) - \frac{c_2L_H}{\tau_I} - \frac{c_2^2}{\tau_I^2}\end{bmatrix}.$$

By Assumption 3, one can then write the following:

$$\begin{aligned}\mathcal{V}_{k+1} - \mathcal{V}_k \leq & -\frac{\lambda_1}{\beta_2}(W_k + h_k) - \lambda_1\|\tilde{u}_k\|^2 - \lambda_1\|\Psi_{1,k}\|^2 \\ & + \frac{c_1L_H}{\sqrt{\beta_1}\tau_I}W_k + \frac{c_1}{\tau_I}\|\tilde{u}_k\| + D\|\Psi_{1,k}\| + \frac{c_2^2}{\tau_I^2} \\ \leq & -\lambda_2\mathcal{V}_k - \lambda_1\|\Psi_{1,k}\|^2 + \beta_3\sqrt{\mathcal{V}_k} + D\|\Psi_{1,k}\| + \frac{c_2^2}{\tau_I^2}\end{aligned}$$

where $\lambda_2 = \min\left[\frac{\lambda_1}{\beta_2}, \lambda_1\right]$ and $\beta_3 = \max\left[\frac{c_1L_H}{\tau_I}\frac{1}{\sqrt{\beta_1}}, \sqrt{2}\frac{c_1}{\tau_I}\right]$.

Thus we see that the closed-loop signals $\|\Psi_{1,k}\|$, $\|\tilde{u}_k\|$ and $\|x - \pi(\hat{u}_k)\|$ of the proposed ESC signals enter a neighbourhood of the origin whose magnitude depends on the magnitude of $\|d_k\|$. This neighbourhood will be of order $\mathcal{O}\left(\frac{c_1^2}{\tau_I^2}\right)$ and $\mathcal{O}\left(\frac{D}{\lambda_1}\right)$.

As $\mathcal{V}_k$ enters a neighbourhood of the origin, it follows that the closed-loop signals enter a neighbourhood of the optimum steady-state conditions (x^*, u^*). This completes the proof. ∎

Remark 1: The proof provides some nominal tuning guidelines for k_g and τ_I. If one fixes τ_I, the analysis suggests to pick $k_g = 1/\tau_I^2$. However, it is clear that there is much more freedom to pick k_g. To demonstrate, assume that one can pick τ_I large enough and a k_g^* such that for every

$k_g < k_g^*$ one obtains:

$$\lim_{\tau_I \to \infty} (\mathcal{V}_{k+1} - \mathcal{V}_k) \leq -\lambda_3 \|x - \pi(\hat{u}_k)\| - \lambda_3 \|\Psi_{1,k}\|^2 + (k_g^* c_1 + D)\|\Psi_{1,k}\|$$

The closed-loop signals will asymptotically enter a neighbourhood of the origin given by:

$$\Omega_{k_g} = \left\{ x \in \mathcal{D}(\hat{u})\, \hat{u} \in \mathcal{U} \;\middle|\; \|\Psi_{1,k}\| \leq \frac{(k_g^* c_1 + D)}{\lambda_3} \right\}$$

Thus, one can establish a maximum gain k_g^* that retains closed-loop stability in the absence of integral action.

IV. Simulation

In this section, we consider the application of the PI-ESC approach to the following nonlinear discrete-time system:

$$x_{k+1} = 0.99 x_k + (u_k - 0.1)(1 + \frac{1}{2}\sin(x_k))$$
$$y_k = 1 + 0.2(x_k - 1)^2$$

The optimum occurs at $x^* = 1$, $u^* = 0.1069$. The PIESC is used with a gain of $k_g = 0.75$ and integral time constant $\tau_I = 50$. The dither signal is $d_k = 0.05\sin(k)$. The estimation gates are set to $K = 0.0001$, $\alpha = 0.01$. The projection algorithm enforces a region where $\|\hat{\theta}_k\| \leq 0.1$. The simulation results are shown in Figure 1. The figure shows the cost function, y_k, the input, u_k, and the integration variable $\hat{u}_k$. The PIESC very effectively converges to the optimum equilibrium conditions. For the sake of comparison, we also compare the performance of the proposed ESC with the perturbation based discrete-time ESC algorithm proposed in [8] given by:

$$\xi_{k+1} = -h_\ell \xi_k + y_k$$
$$\hat{u}_{k+1} = \hat{u}_k - \gamma\alpha\cos(\omega k)(y1 - (1 + h_\ell)\xi_{k+1})$$
$$u_k = \hat{u}_k + \alpha\cos(\omega(k+1))$$

The tuning parameters for the perturbation ESC are $h_\ell = 0.2$, $\gamma = 5/\alpha$, $\alpha = 0.1$, $\omega = 2$. The corresponding ESC performance is shown as the dashed line in Figure 1. As expected, the proposed PIESC provides a faster convergence to the optimum conditions.

V. Conclusion

This paper proposes a proportional-integral extremum-seeking control technique for a class of discrete-time nonlinear dynamical systems with unknown dynamics. The main contribution of this technique is the minimization of the impact of time-scale separation on the transient performance of the extremum-seeking control system in discrete-time.

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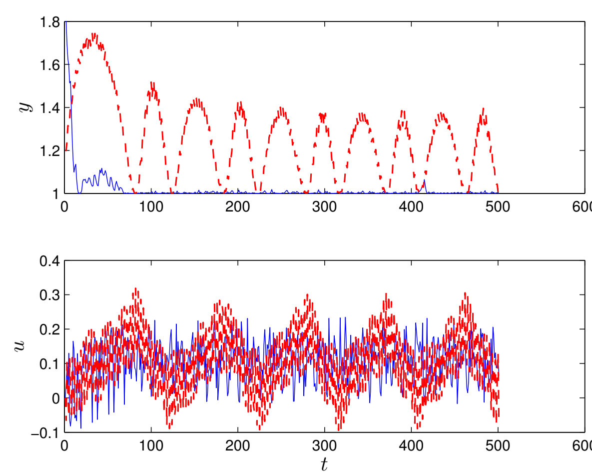

Fig. 1. Performance of the PI-ESC for Example 1 with $d_k = sin(k)$. The upper plot shows the cost function, the middle plot shows the input variable and the bottom, $\hat{u}$, as a function sampling steps k.