A proportional integral extremum-seeking control approach for discrete-time nonlinear systems

Guay, M.; Burns, D.J.

TR2016-065 May 2016

Abstract

This paper proposes a proportional-integral extremum-seeking control technique for a class of discrete-time nonlinear dynamical systems with unknown dynamics. The technique is a generalization of existing time-varying extremum-seeking control techniques that provides fast transient performance of the closed-loop system to the optimum equilibrium of a measured objective function. The main contribution of the proposed technique is the addition of a proportional action that can be used to minimize the impact of a time-scale separation on the transient performance of the extremum-seeking control system. The integral action fulfills the role of standard ESC techniques to identify optimal equilibrium conditions. The effectiveness of the proposed approach is demonstrated using a simulation example.

International Journal of Control

A proportional integral extremum-seeking control approach for discrete-time nonlinear systems

Martin Guay* and Daniel J. Burns †

September 1, 2016

Abstract

This paper proposes a proportional-integral extremum-seeking control technique for a class of discrete-time nonlinear dynamical systems with unknown dynamics. The technique is a generalization of existing time-varying extremum-seeking control techniques that provides fast transient performance of the closed-loop system to the optimum equilibrium of a measured objective function. The main contribution of the proposed technique is the addition of a proportional action that can be used to minimize the impact of a time-scale separation on the transient performance of the extremum-seeking control system. The integral action fulfills the role of standard ESC techniques to identify optimal equilibrium conditions. The effectiveness of the proposed approach is demonstrated using a simulation example.

1 Introduction

Extremum-seeking control (ESC) has grown to become the leading approach to solve real-time optimization problems [16]. Following the seminal work of Krstic and coworkers ([9], [8], [2], [1], [3], [18]), this strikingly general and practically relevant control approach is equipped with an established and well understood control theoretical framework. The main drawback of ESC is the lack of transient performance guarantees. As highlighted in the proof of Krstic and Wang [9], the stability analysis relies on two components: an averaging analysis of the persistently perturbed ESC loop and a time-scale separation of ESC closed-loop dynamics between the fast transients of the system dynamics and the slow quasi steady-state extremum-seeking task. While the averaging

*M. Guay is with the Department of Chemical Engineering, Queens' University, Kingston, ON, Canada. martin.guay@chee.queensu.ca

†D. J. Burns is with the Mechatronics Group at Mitsubishi Electric Research Laboratories, 201 Broadway, Cambridge, MA 02139. burns@merl.com, and is the author to whom correspondence should be addressed.

analysis highlights the stability properties of ESC systems, the need for a slower time-scale for the optimization dynamics invariantly leads to a slow performance of the closed-loop ESC system. The objective of this study is to develop an ESC technique that minimizes the impact of time-scale separation on the transient performance of ESC systems for a class of discrete-time nonlinear dynamical systems.

The vast majority of existing results on ESC have focussed on continuous-time systems. Although discrete-time systems can be treated in an essentially similar fashion, the application of gradient descent in a discrete-time setting requires some care. A discrete-time version of the standard ESC loop was studied in [2] and [3] where convergence results similar to continuous time systems are obtained. A similar algorithm was also proposed in [7] for the tuning of PID controllers in unknown dynamical systems using ESC. Discrete-time ESC subject to stochastic perturbations is studied in [12]. A stochastic ESC approach for a class of discrete-time nonlinear systems is proposed in [11]. The use of approximate parameterizations of the unknown cost function using quadratic functions was recently proposed in [14]. An alternative ESC-like approach was proposed in [17]. In this study, a trajectory based approach is used to analyze the properties of nonlinear optimization algorithms as dynamical systems. It is shown that properties of the nonlinear-optimization algorithms are suitable to assess the convergence of certain classes of ESC applied in a sampled-data approach. This approach was recently studied in the context of global sampling methods in [13] where trajectory based properties of nonlinear optimization methods are used to establish robust convergence. The main objectives with the trajectory based techniques is to analyze the properties of optimization algorithms assuming that they can converge to the true optimum using only the measurement of the objective function and possibly the constraints. In the context of ESC, one must either imply that the nonlinear optimization techniques do not rely on gradient information or, if they do, this gradient must be either measured or estimated. Some techniques such as [19] and [20] make use of sporadic gradient measurements in extremum seeking control. Other techniques [15] go as far as requiring the existence of multiple (nearly) identical systems to enable the estimation of gradient information.

This paper proposes the design of a fast ESC for discrete-time systems. The approach is based on a proportional-integral ESC (PIESC) design technique initially proposed in [6]. The approach extends the time-varying discrete-time ESC technique proposed in [5]. The PIESC technique proposed here is a combination of an integral action which corresponds to the standard ESC control task used to identify the steady-state optimum and a proportional control action designed to ensure that the measured cost function can be optimized instantaneously. Under suitable assumption on the dynamics of the system and the cost function, this action can be shown to minimize the cost over short times while reaching the optimum steady-state conditions. The use of the proportional action is one aspect of the proposed approach that can be used to expand the range of applicability of ESC. One can argue that a large class of problems could be solved by first designing a robustly stabilizing feedback to the unknown system that is amenable to the application of standard ESC. However, the combination of standard ESC with a stabilizing feedback implies considerable *a priori* knowledge of the process dynamics. Precise knowledge of the process dynamics violates the main assumption of ESC that the mathematical formulation of the process is unknown. Furthermore, this study establishes that the proposed ESC can be used as a possible candidate for stabilization (and optimization) of nonlinear discrete-time systems. The corresponding feedback solution can be argued as a Jurdjevic-Quinn damping feedback for discrete-time nonlinear systems, as proposed in [10]. Such state-feedback solutions have not been established in the context of discrete-time ESC design.

The paper is organized as follows. A problem description of the ESC problem along with the key assumptions is given in Section 2. The proposed proportional-integral ESC controller is described in Section 3. The closed-loop stability of the PI-ESC and the main theorem of this study is presented in Section 4. Simulation examples are presented in Section 5 followed by brief conclusions and proposed future work in Section 6.

2 Problem description

We consider a class of nonlinear systems of the form:

$$x_{k+1} = x_k + f(x_k) + g(x_k)u_k \quad (1)$$

$$y_k = h(x_k) \quad (2)$$

where $x_k \in \mathbb{R}^n$ is the vector of state variables at time k, u_k is the vector of input variables at time k taking values in $\mathcal{U} \subset \mathbb{R}^p$ and $y_k \in \mathbb{R}$ is the objective function at step k, to be minimized. It is assumed that $f(x_k)$ and $g(x_k)$ are smooth vector valued functions and that $h(x_k)$ is a smooth function.

The objective is to steer the system to the equilibrium x^* and u^* that achieves the minimum value of $y(= h(x^*))$. The equilibrium (or steady-state) map is the n dimensional vector $x = \pi(u)$ that solves the following equation:

$$f(\pi(u)) + g(\pi(u))u = 0.$$

The corresponding equilibrium cost function is given by:

$$y = h(\pi(u)) = \ell(u) \quad (3)$$

At equilibrium, the problem is reduced to finding the minimizer u^* of $y = \ell(u^*)$. In the following, we let $\mathcal{D}(u)$ represent a neighbourhood of the equilibrium $x = \pi(u)$.

The following additional assumption concerning the steady-state cost function $\ell(u)$ is required.

Assumption 1 *The nonlinear system is such that*

$$\nabla_x h(\pi(u))g(\pi(u))(u - u^*) \geq \alpha_u \|u - u^*\|^2$$

for some positive constant α_u $\forall u \in \mathcal{U}$.

Assumption 2 *The cost $h(x)$ is such that*

1. $\frac{\partial h(x^*)}{\partial x} = 0$
2. $\frac{\partial^2 h(x)}{\partial x \partial x^T} > \beta I, \ \forall x \in \mathbb{R}^n$

where β is a strictly positive constant.

It is assumed that the cost function dynamics has relative degree one in $\mathcal{D}(u)$. The cost function dynamics are expressed as follows. We let $\alpha(x_k, \hat{u}_k) = x_k + f(x_k) + g(x_k)\hat{u}_k$ where $\hat{u}_k$ is used as an estimate of the unknown optimum equilibrium p dimensional vector of input variables, u^*. The rate of change of the cost function $y_k = h(x_{k+1})$ is given by:

$$\begin{aligned} h(x_{k+1}) - h(x_k) = h(x_k + f(x_k) + g(x_k)u_k) \\ - h(\alpha(x_k, \hat{u}_k)) + h(\alpha(x_k, \hat{u}_k)) - h(x_k). \end{aligned}$$

The first two terms can be rewritten using the second order Taylor formula as:

$$\begin{aligned} h(x_k + f(x_k) + g(x_k)u_k) - h(\alpha(x_k, \hat{u}_k)) =& \nabla h(\alpha(x_k, \hat{u}_k))g(x_k)(u_k - \hat{u}_k) \\ &+ \frac{1}{2}(u_k - \hat{u}_k)^\top g(x_k)^\top \nabla^2 h(\tilde{y}_k) g(x_k)(u_k - \hat{u}_k) \end{aligned} \tag{4}$$

where $\tilde{y}_k = \alpha(x_k, \hat{u}_k) + \theta g(x_k)(u_k - \hat{u}_k)$ for $\theta \in (0, 1)$. We rewrite (4) as follows:

$$h(x_k + f(x_k) + g(x_k)u_k) - h(\alpha(x_k)) = \Psi_{1,k}(x_k, u_k, \hat{u}_k)(u_k - \hat{u}_k) \tag{5}$$

where

$$\Psi_{1,k}(x_k, u_k, \hat{u}_k) = (\nabla h(\alpha(x_k, \hat{u}_k))g(x_k) + \frac{1}{2}(u_k - \hat{u}_k)^\top g(x_k)^\top \nabla^2 h(\tilde{y}_k) g(x_k).$$

We also define the following

$$\Psi_{0,k}(x_k, \hat{u}_k) = h(\alpha(x_k, \hat{u}_k)) - h(x_k).$$

and write the cost dynamics as:

$$y_{k+1} - y_k = \Psi_{0,k}(x_k, \hat{u}_k) + \Psi_{1,k}(x_k, u_k, \hat{u}_k)(u_k - \hat{u}_k).$$

The last equation provides a parameterization of the discrete-time cost dynamics that is amenable to the statement of assumptions concerning their stabilizability. The term $\Psi_{0,k}$ identifies the drift termof the unknown dynamics while $\Psi_{1,k}$ provides a representation of the control direction at step k.

By the relative order one assumption on $h(x)$, the system's dynamics can be decomposed and written as:

$$\xi_{k+1} = \xi_k + \psi(\xi_k, y_k) \tag{6}$$

$$y_{k+1} = y_k + \Psi_{0,k}(x_k, \hat{u}_k) + \Psi_{1,k}(x_k, u_k, \hat{u}_k)(u_k - \hat{u}_k) \tag{7}$$

where $\xi_k \in \mathbb{R}^{n-1}$ and $\psi(\xi_k, y_k)$ is a smooth vector valued function. In the process dynamics (6), the variables ξ_k represent the state variables of the zero dynamics of the control system.

The following assumptions provide conditions for the stabilizability of the discrete-time nonlinear systems. The first assumption defines the type of state feedback controllers that are considered.

Assumption 3 *There exists a function $u_k = \alpha_F(x_k, \hat{u}_k)$ that solves the identity:*

$$\alpha_F(x_k, \hat{u}_k) = -k_g \Psi_{1,k}(x_k, \alpha_F(x_k, \hat{u}_k), \hat{u}_k)^T + \hat{u}_k.$$

This assumption simply establishes that the feedback:

$$u_k = -k_g \Psi_{1,k}(x_k, u_k, \hat{u}_k)^T + \hat{u}_k$$

is well defined.

Assumption 4 *There exists a positive definite function $W(\xi)$ that satisfies the following inequalities:*

$$\beta_1 \|x_k - \pi(\hat{u})\|^2 \leq W(\xi) + h(x) \leq \beta_2 \|x_k - \pi(\hat{u})\|^2$$

with positive constants β_1 and β_2, and a positive constant k_g^* such that:

$$W(\xi_{k+1}) + h(\alpha(x_k)) - W(\xi_k) - h(x_k) - k_g^* \|\Psi_{1,k}(x_k, \alpha_F(x_k, \hat{u}_k), \hat{u}_k)\|^2 \\ \leq -\alpha_e \|x_k - \pi(\hat{u}_k)\|^2$$

with positive constant α_e, $\forall x_k \in \mathcal{D}(\hat{u})$ and $\forall \hat{u}_k \in \mathcal{U}$.

Assumption 4 states that $W + h$ is non-increasing along the vector field $f(x) + g(x)u$ over some neighbourhood of the steady-state manifold $x = \pi(u)$ at a fixed value of the input $\hat{u}_k$.

3 Proportional-Integral Perturbation Discrete-time ESC

In this section, we present the proposed ESC controller.

Recall that the cost function dynamics can be parameterized as follows:

$$y_{k+1} = y_k + \theta_{0,k} + \theta_{1,k}^T (u_k - \hat{u}_k)$$

where the time-varying parameters $\theta_{0,k}$ and $\theta_{1,k}$ are identified with $\theta_{0,k} = \Psi_{0,k}$ and $\theta_{1,k} = \Psi_{1,k}^T$.

Since the parameters $\theta_{0,k}$ and $\theta_{1,k}$ are unknown, they must be estimated. Let $\hat{\theta}_{0,k}$ and $\hat{\theta}_{1,k}$ denote the estimates of $\theta_{0,k}$ and $\theta_{1,k}$, respectively. The proposed proportional-integral extremum-seeking controller is given by:

$$u_k = -k_g\hat{\theta}_{1,k} + \hat{u}_k + d_k \tag{8}$$
$$\hat{u}_{k+1} = \hat{u}_k - \frac{1}{\tau_I}\hat{\theta}_{1,k}.$$

where k_g and τ_I are positive constants to be assigned. The term d_k is a dither signal used to provide a sufficiently signal in closed-loop. The dither signal is bounded and such that $\|d_k\| \leq D$ where D is a known positive constant.

In practice, this algorithm can be assigned in the velocity form as follows:

$$u_{k+1} = u_k - k_g(\hat{\theta}_{1,k+1} - \hat{\theta}_{1,k}) - \frac{1}{\tau_I}\hat{\theta}_{1,k} + d_k.$$

In what follows, the analysis will be performed for the controller (8).

3.1 Time-varying parameter estimation approach

This section describes a scheme that allows the accurate estimation of the parameters $\theta_{0,k}$ and $\theta_{1,k}$. Note that the estimation of $\theta_{0,k}$ is necessary to ensure that the estimates of $\theta_{1,k}$ are not biased.

Consider the following state predictor

$$\hat{y}_{k+1} \quad = \quad \hat{y}_k + \hat{\theta}_{0,k} + \hat{\theta}_{1,k}^T(u_k - \hat{u}_k) + K_k e_k - \omega_{k+1}^T(\hat{\theta}_k - \hat{\theta}_{k+1}) \tag{9}$$

where $\hat{\theta}_k = [\hat{\theta}_{0,k}, \hat{\theta}_{1,k}^T]^T$ is the vector of parameter estimates at time step k given by any update law, K_k is a correction factor at time step k, $e_k = y_k - \hat{y}_k$ is the state estimation error at time step k. We let $\phi_k = [1, (u_k - \hat{u}_k)^T]^T$. The variable ω_k is the following output filter at time step k

$$w_{k+1} = w_k + \phi_k - K_k w_k, \tag{10}$$

with $\omega_0 = 0$. In what follows, we denote the parameter estimation error as $\tilde{\theta}_k = \theta_k - \hat{\theta}_k$.

Using the state predictor defined in (9) and the output filter defined in (10), the prediction error $e_k = y_k - \hat{y}_k$ is given by

$$\begin{aligned} e_{k+1} &= e_k + \phi_k\tilde{\theta}_{k+1} - K_k e_k + \omega_{k+1}^T(\hat{\theta}_k - \hat{\theta}_{k+1}) + \omega_{k+1}^T(\theta_{k+1} - \theta_k) \\ e_0 &= y_0 - \hat{y}_0. \end{aligned} \tag{11}$$

An auxiliary variable η_k is introduced which is defined as $\eta_k = e_k - \omega_k^T\tilde{\theta}_k$. Its dynamics are described as follows

$$\begin{aligned} \eta_{k+1} &= \eta_k - K\eta_k + \omega_{k+1}^T(\theta_{k+1} - \theta_k) = \eta_k - K\eta_k + \vartheta_k \\ \eta_0 &= e_0. \end{aligned} \tag{12}$$

Since ϑ_k is unknown, it is necessary to use an estimate, $\hat{\eta}$, of η . The estimate is generated by the recursion:

$$\hat{\eta}_{k+1} = \hat{\eta}_k - K_k \hat{\eta}_k. \tag{13}$$

The resulting dynamics of the η estimation error are:

$$\tilde{\eta}_{k+1} = \tilde{\eta}_k - K_k \tilde{\eta}_k + \omega_{k+1}^T (\theta_{k+1} - \theta_k) \tag{14}$$

The proposed parameter estimation routine is an extension of recursive least squares such as presented in [4] for the estimation of time-varying parameters. Let the identifier matrix Σ_k be defined as

$$\Sigma_{k+1} = \alpha \Sigma_k + \omega_k \omega_k^T + \sigma I, \quad \Sigma_0 = \alpha I \succ 0 \tag{15}$$

with an inverse generated by the recursion

$$\Sigma_{k+1}^{-1} = (\alpha \Sigma_k + \sigma I)^{-1} - (\alpha \Sigma_k + \sigma I)^{-1} \omega_k Q_k \omega_k^T (\alpha \Sigma_k + \sigma I)^{-1} \tag{16}$$

where $Q_k = (1 + \frac{1}{\alpha} \omega_k^T (\alpha \Sigma_k + \sigma I)^{-1} \omega_k)^{-1}$. Using equations (9), (10), and (13), it follows from standard arguments ([4]) that the preferred parameter update law is given by:

$$\hat{\theta}_{k+1} = \hat{\theta}_k + (\alpha \Sigma_k + \sigma I)^{-1} \omega_k Q_k (e_k - \hat{\eta}_k) \tag{17}$$

To ensure that the parameter estimates remain within the constraint set Θ_k, we propose to use a projection operator of the form:

$$\bar{\bar{\theta}}_{k+1} = \text{Proj}\{\hat{\theta}_k + (\alpha \Sigma_k + \sigma I)^{-1} \omega_k Q_k (e_k - \hat{\eta}_k), \Theta_k\} \tag{18}$$

The operator Proj represents an orthogonal projection onto the surface of the uncertainty set applied to the parameter estimate.The parameter uncertainty set is defined by the ball function $B(\hat{\theta}_c, z_{\hat{\theta}c})$, where $\hat{\theta}_c$ and $z_{\hat{\theta}c}$ are the parameter estimate and set radius found at the latest set update.

Following [4], the projection operator is designed such that

- $\hat{\theta}_{k+1} \in \Theta_0$
- $\bar{\bar{\theta}}_{k+1}^T \Sigma_{k+1} \bar{\bar{\theta}}_{k+1} \leq \tilde{\theta}_{k+1}^T \Sigma_{k+1} \tilde{\theta}_{k+1}$

One possible algorithm for the projection algorithm is as follows. Define the upper bound for $\|\theta\|$ $(= L_1)$. Let $R =$Chol(Σ_{k+1}) denote the Cholesky factor of Σ_{k+1}. Then we perform the following:

Algorithm 1 *If* $\|\hat{\theta}_{k+1}\| \geq L_1$ *then*

- *Let* $\delta = \frac{L_1 \hat{\theta}_{k+1}}{\|\hat{\theta}_{k+1}\|}$,
- *Let* $z_\rho = \sqrt{\delta^T \Sigma_{k+1} \delta}$,

- *With* $\rho = R\hat{\theta}_{k+1}$ *define* $\bar{\rho} = \frac{\rho z_\rho}{\|\rho\|}$,

- *Let* $\bar{\hat{\theta}}_{k+1} = R^{-1}\bar{\rho}$.

Otherwise,

- *Let* $\bar{\hat{\theta}}_{k+1} = \hat{\theta}_{k+1}$.

It is assumed that the trajectories of the system are such that the following condition is met.

Assumption 5 *[4] There exists constants* $\beta_T > 0$ *and* $T > 0$ *such that*

$$\frac{1}{T}\sum_{i=k}^{k+T-1} \omega_i \omega_i^T > \beta_T I, \ \forall k > T. \tag{19}$$

This requirement is a standard persistency of excitation condition that can be found in most references on adaptive control and adaptive estimation. The reader is referred to [4] for more details.

3.2 Summary of the approach

The key elements of the approach can be summarized schematically in Figure 1. The technique combines a time-varying estimation algorithm with gains K and α and the PI-ESC algorithm with proportional gain k_g and integral constant τ_I.

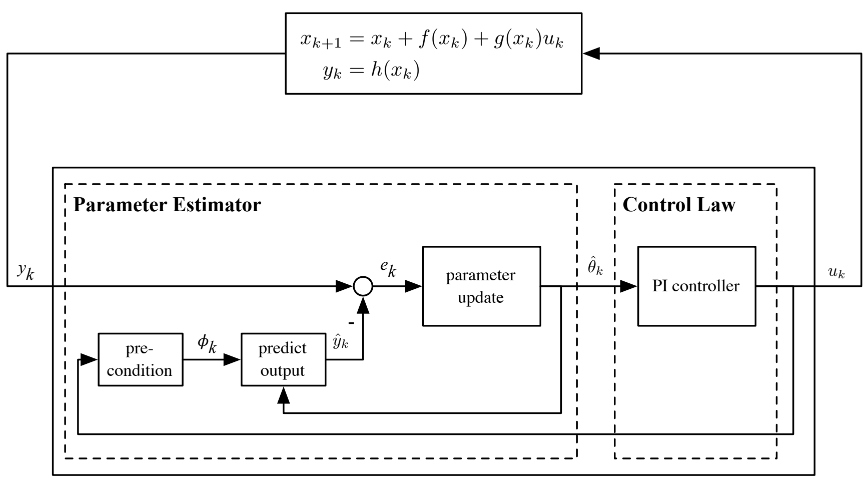

Figure 1: Schematic representation of the PI-ESC approach.

A detailed schematic description of the closed-loop PI-ESC system is provided in Figure 2.

The tuning of the constant and the performance of the closed-loop system are addressed in the next section.

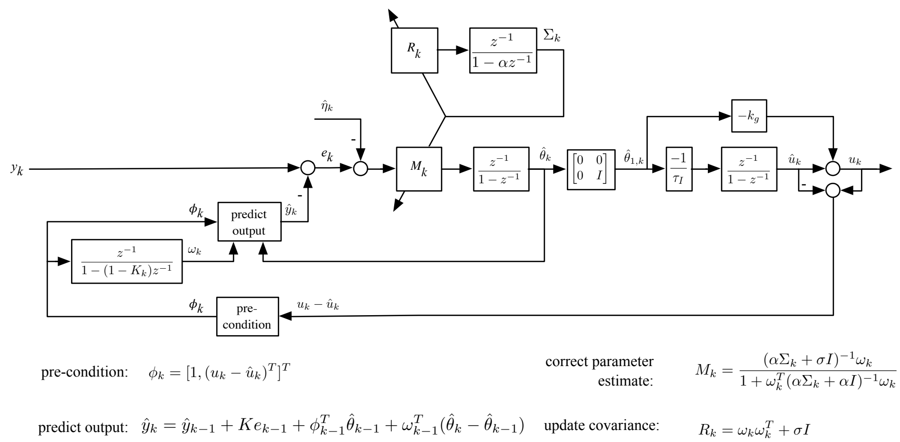

Figure 2: Schematic representation of the PI-ESC approach.

4 Closed-loop stability of PI-ESC

In this section, we present the main result of this study. It is stated in the form of the following theorem.

Theorem 1 *Consider the nonlinear discrete-time system* (1) *with cost function* (2)*, the extremum seeking controller* (8) *and parameter estimation scheme* (9)*,* (10)*,* (13)*,* (15) *and* (18)*. Let Assumptions 1-5 be fulfilled. Then there exists positive constants α, K, k_g and τ_I such that for every $\tau_I \geq \tau_I^*$, the states x_k and input u_k of the closed-loop system enter a neighbourhood of the unknown optimum (x^*, u^*).*

Proof: Let $\tilde{u}_k = u_k - u^*$ and consider the Lyapunov function:

$$W_k = \tilde{\theta}_k^T \Sigma_k \tilde{\theta}_k.$$

Consider the following:

$$\begin{aligned} W_{k+1} - W_k &= \tilde{\bar{\theta}}_{k+1}^T \Sigma_{k+1} \tilde{\bar{\theta}}_{k+1} - \tilde{\theta}_k^T \Sigma_k \tilde{\theta}_k \\ &\leq \tilde{\theta}_{k+1}^T \Sigma_{k+1} \tilde{\theta}_{k+1} - \tilde{\theta}_k^T \Sigma_k \tilde{\theta}_k. \end{aligned} \tag{20}$$

where the final inequality arises as a result of the properties of the projection algorithm.

The parameter estimation error dynamics is given by:

$$\begin{aligned} \tilde{\theta}_{k+1} &= \tilde{\theta}_k + (\theta_{k+1} - \theta_k) - (\alpha\Sigma_k + \sigma I)^{-1}\omega_k Q_k(e_k - \hat{\eta}_k). \\ &= \tilde{\theta}_k + (\theta_{k+1} - \theta_k) - (\alpha\Sigma_k + \sigma I)^{-1}\omega_k Q_k \omega_k^T \tilde{\theta}_k \\ &\quad - (\alpha\Sigma_k + \sigma I)^{-1}\omega_k Q_k \tilde{\eta}_k \end{aligned}$$

Note that by construction one can write the parameter estimation error dynamics as follows:

$$\tilde{\theta}_{k+1} = (\theta_{k+1} - \theta_k) + \Sigma_{k+1}^{-1}(\alpha\Sigma_k + \sigma I)\tilde{\theta}_k - (\alpha\Sigma_k + \sigma I)^{-1}\omega_k Q_k \tilde{\eta}_k$$

In the following, we define $v_k = (\theta_{k+1} - \theta_k) - (\alpha\Sigma_k + \sigma I)^{-1}\omega_k Q_k \tilde{\eta}_k$. Upon substitution of the dynamics of $\tilde{\theta}_k$, one obtains:

$$\tilde{\theta}_{k+1} = \Sigma_{k+1}^{-1}(\alpha\Sigma_k + \sigma I)\left[\Sigma_k^{-1}(\alpha\Sigma_{k-1} + \sigma I)\tilde{\theta}_{k-1} + v_{k-1}\right] + v_k \tag{21}$$

One obtains the following by induction:

$$\tilde{\theta}_{k+1} = \prod_{i=0}^{k}\left[\Sigma_{k-i+1}^{-1}(\alpha\Sigma_{k-i} + \sigma I)\right]\tilde{\theta}_0 + \sum_{i=0}^{k}\prod_{j=0}^{i}\left[\Sigma_{k-j+1}^{-1}(\alpha\Sigma_{k-j} + \sigma I)\right]v_{k-i}$$

where we apply the convention $\prod_{j=0}^{0}\left[\Sigma_{k-j+1}^{-1}(\alpha\Sigma_{k-j} + \sigma I)\right] = 1$.

$$\begin{aligned}
\tilde{\theta}_{k+1} =& \Sigma_{k+1}^{-1}\prod_{i=1}^{k}\left[(\alpha\Sigma_{k-i} + \sigma I)\Sigma_{k-i}^{-1}\right](\alpha\Sigma_0 + \sigma I)\tilde{\theta}_0 + \sum_{i=0}^{k}\prod_{j=0}^{i}\left[\Sigma_{k-j+1}^{-1}(\alpha\Sigma_{k-j} + \sigma I)\right]v_{k-i} \\
=& \Sigma_{k+1}^{-1}\prod_{i=1}^{k-1}\left[(\alpha I + \sigma\Sigma_{k-i}^{-1})\right](\alpha\Sigma_0 + \sigma I)\tilde{\theta}_0 + \sum_{i=0}^{k}\prod_{j=0}^{i}\left[\Sigma_{k-j+1}^{-1}(\alpha\Sigma_{k-j} + \sigma I)\right]v_{k-i}
\end{aligned}$$

The matrix Σ_{k+1} can be bounded as follows. The recursion for Σ_k can be rewritten as:

$$\Sigma_{k+1} = \alpha^{k+1}\Sigma_0 + \sum_{i=0}^{k}\alpha^{k-i}\omega_i\omega_i^T + \sum_{i=0}^{k}\alpha^{k-i}\sigma I.$$

Then one can write:

$$\begin{aligned}
\Sigma_{k+1} &\leq \alpha^{k+1}\Sigma_0 + \sum_{i=0}^{k}\alpha^{k-i}\left[\sum_{j=1}^{T}\omega_{i+j}\omega_{i+j}^T + \sigma I\right] \\
&\leq \alpha^{k+1}\Sigma_0 + \sum_{i=0}^{k}\alpha^{k-i}T(\beta + \sigma)I \leq \alpha^{k+1}\Sigma_0 + \frac{1-\alpha^{k+1}}{1-\alpha}T(\beta + \sigma)I
\end{aligned}$$

Similarly, one can provide a lower bound for Σ_{k+1}. Consider the quantity:

$$T\Sigma_{k+1} = T\alpha^{k+1}\Sigma_0 + T\sum_{i=0}^{k}\alpha^{k-i}\omega_i\omega_i^T + T\sum_{i=0}^{k}\alpha^{k-i}\sigma I$$

$$\geq T\alpha^{k+1}\Sigma_0 + \sum_{i=T}^{k}\alpha^{k-i}\omega_i\omega_i^T + \sum_{i=T-1}^{k-1}\alpha^{k-i}\omega_i\omega_i^T +$$

$$\ldots + \sum_{i=0}^{k-T}\alpha^{k-i}\omega_i\omega_i^T + T\sum_{i=0}^{k}\alpha^{k-i}\sigma I$$

$$= T\alpha^{k+1}\Sigma_0 + \sum_{i=0}^{k-T}\alpha^{k-i-T}\omega_{i+T}\omega_{i+T}^T$$

$$+ \sum_{i=0}^{k-T}\alpha^{k-i-T-1}\omega_{i+T-1}\omega_{i+T-1}^T +$$

$$\ldots + \sum_{i=0}^{k-T}\alpha^{k-i}\omega_i\omega_i^T + T\sum_{i=0}^{k}\alpha^{k-i}\sigma I$$

$$= T\alpha^{k+1}\Sigma_0 + \sum_{i=0}^{k-T}\alpha^{k-i}\sum_{j=0}^{T-1}\alpha^{-j}\omega_{i+j}\omega_{i+j}^T + T\sum_{i=0}^{k}\alpha^{k-i}\sigma I$$

$$\geq T\alpha^{k+1}\Sigma_0 + \sum_{i=0}^{k-T}\alpha^{k-i}\sum_{j=0}^{T-1}\omega_{i+j}\omega_{i+j}^T + T\sum_{i=0}^{k}\alpha^{k-i}\sigma I$$

$$\geq T\alpha^{k+1}\Sigma_0 + \sum_{i=0}^{k-N}\alpha^{k-i}T\beta_T I + T\sum_{i=0}^{k}\alpha^{k-i}\sigma I$$

$$\geq T\alpha^{k+1}\Sigma_0 + \frac{\alpha^T - \alpha^{k+1}}{1-\alpha}T\beta_T I + T\sigma I$$

$$= T\alpha^{k+1}\Sigma_0 + \frac{\alpha^T(1-\alpha^{k-T+1}}{1-\alpha}T\beta_T + T\sigma I$$

$$\geq T\alpha^{k+1}\Sigma_0 + \frac{\alpha^T}{1-\alpha}T\beta_T I + T\sigma I \geq \frac{\alpha^T}{1-\alpha}T\beta_T I + T\sigma I.$$

Assuming that $\Sigma_0 = \alpha_0 I$, one gets the following bounds:

$$\frac{\alpha^T}{1-\alpha}\beta_T I + \sigma I \leq \Sigma_{k+1} \leq \alpha_0 I + \frac{1}{1-\alpha}T(\beta+\sigma)I.$$

or,

$$\frac{1-\alpha}{\alpha_0(1-\alpha)+T(\beta+\sigma)} \leq \Sigma_{k+1}^{-1} \leq \frac{1-\alpha}{\beta_T\alpha^T+\sigma(1-\alpha)}I. \tag{22}$$

By the dynamics of $\tilde{\eta}_k$, it is easy to show that:

$$\tilde{\eta}_{k+1} = \sum_{i=1}^{k}(1-K)^{k-i+1}\tilde{\eta}_0 + \sum_{i=1}^{k}(1-K)^{k-i}\omega_i^T(\theta_{i+1}-\theta_i)$$

The term v_k can be written as:

$$\begin{aligned} v_k =& (\theta_{k+1}-\theta_k) - (\alpha\Sigma_k + \sigma I)^{-1}\omega_k Q_k \tilde{\eta}_k \\ =& (\theta_{k+1}-\theta_k) - (\alpha\Sigma_k + \sigma I)^{-1}\omega_k Q_k \left(\sum_{i=1}^{k-1}(1-K)^{k-i+1}\tilde{\eta}_0 + \sum_{i=1}^{k-1}(1-K)^{k-i}\omega_i^T(\theta_{i+1}-\theta_i)\right) \end{aligned}$$

As a result, one obtains the upper bound:

$$\begin{aligned} \|\tilde{\eta}_{k+1}\| =& \sum_{i=1}^{k}(1-K)^{k-i+1}\|\tilde{\eta}_0\| \\ &+ \sum_{i=1}^{k}(1-K)^{k-i}\|\omega_i\|\|(\theta_{i+1}-\theta_i)\| \\ \le& \sum_{i=1}^{k}(1-K)^{k-i+1}\|\tilde{\eta}_0\| \\ &+ \sum_{i=1}^{k}(1-K)^{k-i}\sqrt{\beta}\|\theta_{i+1}-\theta_i\| \end{aligned}$$

The parameter estimation error is such that:

$$\begin{aligned} \|\tilde{\theta}_{k+1}\| \le& \prod_{i=0}^{k}\left(\alpha\|\Sigma_{k+1-i}^{-1}\|\|\Sigma_{k-i}\| + \sigma\|\Sigma_{k+1-i}^{-1}\|\right)\|\tilde{\theta}_0\| \\ &+ \sum_{i=1}^{k}\prod_{j=0}^{i}\left(\alpha\|\Sigma_{k-j+1}^{-1}\|\|\Sigma_{k-j}\| + \sigma\|\Sigma_{k-j+1}^{-1}\|\right)\|v_i\| \\ \le& \prod_{i=0}^{k}\left(\alpha\left(\frac{(\alpha_0+\sigma)(1-\alpha)+T(\beta+\sigma)}{\beta_T\alpha^T+\sigma(1-\alpha)}\right)\right)\|\tilde{\theta}_0\| \\ &+ \sum_{i=1}^{k}\prod_{j=0}^{i}\left(\alpha\left(\frac{(\alpha_0+\sigma)(1-\alpha)+T(\beta+\sigma)}{\beta_T\alpha^T+\sigma(1-\alpha)}\right)\right)\|v_i\|. \end{aligned}$$

By smoothness of $\Psi_{0,k}$ and $\Psi_{1,k}$, there exists positive constants L_{Ψ_1}, L_{Ψ_1} and L_{Ψ_1} such that:

$$\begin{aligned} \|\theta_{i+1}-\theta_i\| \le& \|\Psi_{0,i+1}-\Psi_{0,i}\| + \|\Psi_{1,i+1}-\Psi_{1,i}\| \\ \le& L_{\Psi_1}\|x_{k+1}-x_k\| + L_{\Psi_2}\|\hat{u}_{k+1}-\hat{u}_k\| + L_{\Psi_3}\|(u_{k+1}-\hat{u}_{k+1})-(u_k-\hat{u}_k)\| \end{aligned}$$

$\forall x_k \in \mathcal{D}(u)$ and $\forall u \in \mathcal{U}$. Upon substitution of x_{k+1} $\hat{u}_{k+1}$ and u_{k+1}, one obtains:

$$\begin{aligned}\|\theta_{i+1} - \theta_i\| \leq & L_{\Psi_1} \| f(x_k) + g(x_k)(-k_g \hat{\theta}_{1,k} + \hat{u}_k + d_k)\| \\ & + \frac{L_{\Psi_2}}{\tau_I} \|\hat{\theta}_{1,k}\| + k_g L_{\Psi_3} \|\hat{\theta}_{k+1} - \hat{\theta}_k\|.\end{aligned}$$

By smoothness of $f(x)$ and $g(x)$, it follows that there exists positive constants L_F and L_G such that

$$\|f(x_k) - f(\pi(\hat{u}_k))\| \leq L_F \|x_k - \pi(\hat{u}_k)\|, \ \|g(x_k) - g(\pi(\hat{u}_k))\| \leq L_G \|x_k - \pi(\hat{u}_k)\|.$$

As a result, one obtains the following inequality:

$$\begin{aligned}\|\theta_{i+1} - \theta_i\| \leq & L_{\Psi_1} L_F \|x_k - \pi(\hat{u}_k)\| + k_g L_{\Psi_1} L_G \|x_k - \pi(\hat{u}_k)\| \|\hat{\theta}_{1,k}\| + L_{\Psi_1} L_G \|x_k - \pi(\hat{u}_k)\| \|d_k\| \\ & + k_g L_{\Psi_1} \|g(\pi(\hat{u}_k))\| \|\hat{\theta}_{1,k}\| + L_{\Psi_1} \|g(\pi(\hat{u}_k))\| \|d_k\| + \frac{L_{\Psi_2}}{\tau_I} \|\hat{\theta}_{1,k}\| + k_g L_{\Psi_3} \|\hat{\theta}_{k+1} - \hat{\theta}_k\|\end{aligned}$$

Using the bounds $\|d_k\| \leq D$ and $\|\hat{\theta}_k\| \leq L_1$, the following inequality results:

$$\begin{aligned}\|\theta_{i+1} - \theta_i\| \leq & (L_{\Psi_1} L_F + k_g L_{\Psi_1} L_G + D L_{\Psi_1} L_G) \|x_k - \pi(\hat{u}_k)\| \\ & + k_g L_{\Psi_1} G L_1 + D L_{\Psi_1} G + \frac{L_1 L_{\Psi_2}}{\tau_I} + 2 k_g L_{\Psi_3} L_1\end{aligned}$$

or, finally,

$$\|\theta_{i+1} - \theta_i\| \leq b_1(k_g, D) \|x_k - \pi(\hat{u}_k)\| + b_0(k_g, \frac{1}{\tau_I}, D).$$

Without loss of generality, we also assume that $\|\tilde{\eta}_0\| = 0$.

Let us assume that there exists a positive constant β such that:

$$\frac{1}{T} \sum_{j=1} \omega_{k+j} \omega_{k+j}^t \leq \beta I,$$

for all $k > 0$. (By definition, the boundedness of ω_k is guaranteed if u_k and $\hat{u}_k$ are in $\mathcal{U}$.).

Then one can write:

$$\begin{aligned}\|\tilde{\theta}_{k+1}\| \leq & \frac{1-\alpha}{\beta_T} \alpha^{k-T+1} \alpha_0 \|\tilde{\theta}_0\| + \Upsilon(T, \alpha, K) b_1(k_g, D) \|x_k - \pi(\hat{u}_k)\| + \Upsilon(T, \alpha, K) b_0(k_g, \frac{1}{\tau_I}, D) \\ = & c_1 + c_2 \|x_k - \pi(\hat{u}_k)\|\end{aligned}$$

where

$$\begin{aligned}\Upsilon(T, \alpha, K) = & \frac{1 - \alpha^{k+1}}{\beta_T \alpha^T} \alpha_0 + \frac{1 - \alpha^{k+1}}{\beta_T \alpha^T (1-\alpha)} T\beta + \frac{(1-\alpha)(1 - \alpha^{k+1}(1-K)^{k+1})}{(1-K)(\beta_T \alpha^T)^2} \alpha_0 \beta \\ & + \frac{1 - \alpha^{k+1}(1-K)^{k+1}}{(1-K)(\beta_T \alpha^T)^2} T\beta^2.\end{aligned}$$

We thus see that the parameter estimation error will tend to a neighbourhood of the origin. The size of this neighbourhood depends primarily on the constant T associated with the persistency of excitation condition.

As above, we pose the following Lyapunov function candidate:

$$\mathcal{V} = W + h + \frac{1}{2}\tilde{u}^T\tilde{u}.$$

The recursion of $\mathcal{V}$ yields:

$$\begin{aligned}\mathcal{V}_{k+1} - \mathcal{V}_k &= W_{k+1} - W_k + \Psi_{0,k} + \Psi_{1,k}(u_k - \hat{u}_k)\\ &+ \frac{1}{2}\tilde{u}_{k+1}^T\tilde{u}_{k+1} - \frac{1}{2}\tilde{u}_k^\top\tilde{u}_k.\end{aligned}$$

Substitution of the ESC yields:

$$\begin{aligned}\mathcal{V}_{k+1} - \mathcal{V}_k &= W_{k+1} - W_k + \Psi_{0,k} - k_g\Psi_{1,k}\hat{\theta}_{1,k} + \Psi_{1,k}d_k\\ &+ \frac{1}{2}\left(\tilde{u}_k + \frac{1}{\tau_I}\hat{\theta}_{1,k}\right)^T\left(\tilde{u}_k + \frac{1}{\tau_I}\hat{\theta}_{1,k}\right) - \frac{1}{2}\tilde{u}_k^\top\tilde{u}_k.\end{aligned}$$

Replacing $\hat{\theta}_{1,k} = \Psi_{1,k}^T - \tilde{\theta}_{1,k}$ gives:

$$\begin{aligned}\mathcal{V}_{k+1} - \mathcal{V}_k =& W_{k+1} - W_k + \Psi_{0,k} - k_g^*\Psi_{1,k}\Psi_{1,k}^T - (k_g - k_g^*)\Psi_{1,k}\Psi_{1,k}^T\\ &+ k_g\Psi_{1,k}\tilde{\theta}_{1,k} + \Psi_{1,k}d_k + \frac{1}{\tau_I}\tilde{u}_k^\top(\Psi_{1,k}^T - \tilde{\theta}_{1,k})\\ &+ \frac{1}{2\tau_I^2}(\Psi_{1,k}^T - \tilde{\theta}_{1,k})^\top(\Psi_{1,k}^T - \tilde{\theta}_{1,k})\end{aligned}$$

Let $\tilde{k}_g = k_g - k_g^*$. By Assumptions 1 and 4, one obtains:

$$\begin{aligned}\mathcal{V}_{k+1} - \mathcal{V}_k \leq& -\alpha_e\|x - \pi(\hat{u}_k)\|^2 - \left(\tilde{k}_g - \frac{1}{2\tau_I^2}\right)\|\Psi_{1,k}\|^2\\ &+ \left|\left(k_g - \frac{1}{\tau_I^2}\right)\right|\|\Psi_{1,k}\|\|\tilde{\theta}_{1,k}\| + \|\Psi_{1,k}\|\|d_k\|\\ &- \frac{\alpha_u}{\tau_I}\|\tilde{u}_k\|^2 + \frac{L_H}{\tau_I}\|x - \pi(\hat{u}_k)\|\|\tilde{u}_k\| + \frac{1}{\tau_I}\|\tilde{u}_k\|\|\tilde{\theta}_{1,k}\|\\ &+ \frac{1}{2\tau_I^2}\|\tilde{\theta}_{1,k}\|^2\end{aligned}$$

where L_H is the Lipschitz constant associated with

$$\|\Psi_{1,k} - \nabla h(\hat{u}_k)g(\pi(\hat{u}_k))\| \leq L_H\|x - \pi(\hat{u}_k)\|.$$

Substituting for the upper bound of $\|\tilde{\theta}_k\|$, one obtains

$$\begin{aligned}
&\mathcal{V}_{k+1} - \mathcal{V}_k \leq -\alpha_e\|x - \pi(\hat{u}_k)\|^2 - \left(\tilde{k}_g - \frac{1}{2\tau_I^2}\right)\|\Psi_{1,k}\|^2 \\
&+ \left|\left(k_g - \frac{1}{\tau_I^2}\right)\right| c_1\|\Psi_{1,k}\| + D\|\Psi_{1,k}\| \\
&+ \left|\left(k_g - \frac{1}{\tau_I^2}\right)\right| c_2\|\Psi_{1,k}\|\|x - \pi(\hat{u}_k)\| - \frac{\alpha_u}{\tau_I}\|\tilde{u}_k\|^2 \\
&+ \frac{c_1 L_H}{\tau_I}\|x - \pi(\hat{u}_k)\| + \frac{c_2 L_H}{\tau_I}\|x - \pi(\hat{u}_k)\|^2 \\
&+ \frac{c_1}{\tau_I}\|\tilde{u}_k\| + \left(\frac{L_H}{\tau_I} + \frac{c_2}{\tau_I}\right)\|\tilde{u}_k\|\|x - \pi(\hat{u}_k)\| + \frac{c_1^2}{\tau_I^2} + \frac{c_2^2}{\tau_I^2}\|x - \pi(\hat{u}_k)\|^2
\end{aligned}$$

Rearranging and letting $k_g = \frac{1}{\tau_I^2}$, one obtains:

$$\begin{aligned}
\mathcal{V}_{k+1} - \mathcal{V}_k \leq &- \left[\begin{array}{ccc} \|x - \pi(\hat{u}_k)\| & \|\tilde{u}_k\| & \|\Psi_{1,k}\| \end{array}\right] \\
&\times \left[\begin{array}{ccc} \alpha_e - \frac{c_2 L_H}{\tau_I} - \frac{c_2^2}{\tau_I^2} & -\frac{c_2 + L_H}{2\tau_I} & 0 \\ -\frac{c_2 + L_H}{2\tau_I} & \frac{\alpha_u}{\tau_I} & 0 \\ 0 & 0 & \left(\frac{1}{2\tau_I^2}\right) - k_g^* \end{array}\right] \\
&\times \left[\begin{array}{c} \|x - \pi(\hat{u}_k)\| \\ \|\tilde{u}_k\| \\ \|\Psi_{1,k}\| \end{array}\right] \\
&+ \frac{c_1 L_H}{\tau_I}\|x - \pi(\hat{u}_k)\| + \frac{c_1}{\tau_I}\|\tilde{u}_k\| + D\|\Psi_{1,k}\| + \frac{c_1^2}{\tau_I^2}
\end{aligned}$$

It is easy to see that there exists a τ_I^* such that $\forall \tau_I > \tau_I^*$, with $k_g = \frac{1}{\tau_I^2}$ and $k_g^* < \frac{1}{2\tau_I^2}$, the last inequality can be written as:

$$\begin{aligned}
\mathcal{V}_{k+1} - \mathcal{V}_k \leq &- \lambda_1\|x - \pi(\hat{u}_k)\|^2 - \lambda_1\|\tilde{u}_k\|^2 - \lambda_1\|\Psi_{1,k}\|^2 + \frac{c_1 L_H}{\tau_I}\|x - \pi(\hat{u}_k)\| \\
&+ \frac{c_1}{\tau_I}\|\tilde{u}_k\| + D\|\Psi_{1,k}\| + \frac{c_1^2}{\tau_I^2}
\end{aligned}$$

for a positive constant $\lambda_1 > 0$ taken as the minimum eigenvalue of the matrix:

$$\left[\begin{array}{ccc} \alpha_e - \frac{c_2 L_H}{\tau_I} - \frac{c_2^2}{\tau_I^2} & -\frac{c_2 + L_H}{2\tau_I} & 0 \\ -\frac{c_2 + L_H}{2\tau_I} & \frac{\alpha_u}{\tau_I} & 0 \\ 0 & 0 & \left(\frac{1}{2\tau_I^2}\right) - k_g^* \end{array}\right].$$

By Assumption 4, one can then write the following:

$$\begin{aligned}
\mathcal{V}_{k+1} - \mathcal{V}_k &\leq -\frac{\lambda_1}{\beta_2}(W_k + h_k) - \lambda_1\|\tilde{u}_k\|^2 - \lambda_1\|\Psi_{1,k}\|^2 + \frac{c_1 L_H}{\sqrt{\beta_1}\tau_I}W_k + \frac{c_1}{\tau_I}\|\tilde{u}_k\| + D\|\Psi_{1,k}\| + \frac{c_2^2}{\tau_I^2} \\
&\leq -\lambda_2\mathcal{V}_k - \lambda_1\|\Psi_{1,k}\|^2 + \beta_3\sqrt{\mathcal{V}_k} + D\|\Psi_{1,k}\| + \frac{c_2^2}{\tau_I^2}
\end{aligned}$$

where

$$\lambda_2 = \min \left[\frac{\lambda_1}{\beta_2}, \, \lambda_1 \right]$$

and

$$\beta_3 = \max \left[\frac{c_1 L_H}{\tau_I} \frac{1}{\sqrt{\beta_1}}, \, \sqrt{2} \frac{c_1}{\tau_I} \right].$$

Thus we see that the closed-loop signals $\|\Psi_{1,k}\|$, $\|\tilde{u}_k\|$ and $\|x - \pi(\hat{u}_k)\|$ of the proposed ESC signals enter a neighbourhood of the origin whose magnitude depends on the magnitude of $\|d_k\|$. This neighbourhood will be of order $\mathcal{O}\left(\frac{c_1^2}{\tau_I^2}\right)$ and $\mathcal{O}\left(\frac{D}{\lambda_1}\right)$.

As $\mathcal{V}_k$ enters a neighbourhood of the origin, it follows that the closed-loop signals enter a neighbourhood of the optimum steady-state conditions $(x^*, \, u^*)$. This completes the proof. ■

Remark 1 *The proof provides some nominal tuning guidelines for k_g and τ_I. If one fixes τ_I, the analysis suggests to pick $k_g = 1/\tau_I^2$. However, it is clear that there is much more freedom to pick k_g. To demonstrate, assume that one can pick τ_I large enough such that:*

$$\lim_{\tau_I \to \infty} (\mathcal{V}_{k+1} - \mathcal{V}_k) \leq - \left[\begin{array}{cc} \|x - \pi(\hat{u}_k)\| & \|\Psi_{1,k}\| \end{array} \right] \times \left[\begin{array}{cc} \alpha_e & -\frac{k_g c_2}{2} \\ -\frac{k_g c_2}{2} & k_g \end{array} \right] \left[\begin{array}{c} \|x - \pi(\hat{u}_k)\| \\ \|\Psi_{1,k}\| \end{array} \right] + (k_g c_1 + D)\|\Psi_{1,k}\|$$

Consequently, we see that there exists a $\bar{k}_g$ such that for every $k_g < \bar{k}_g$ the inequality can be written as:

$$\begin{aligned} \lim_{\tau_I \to \infty} (\mathcal{V}_{k+1} - \mathcal{V}_k) \leq & - \lambda_3 \|x - \pi(\hat{u}_k)\| - \lambda_3 \|\Psi_{1,k}\|^2 \\ & + (\bar{k}_g c_1 + D)\|\Psi_{1,k}\| \end{aligned}$$

The closed-loop signals will asymptotically enter a neighbourhood of the origin given by:

$$\Omega_{k_g} = \left\{ x \in \mathcal{D}(\hat{u}) \, \hat{u} \in \mathcal{U} \, \middle| \, \|\Psi_{1,k}\| \leq \frac{(\bar{k}_g c_1 + D)}{\lambda_3} \right\}$$

Thus, one can establish a maximum gain $\bar{k}_g$ that retains closed-loop stability in the absence of integral action. Moreover, closed-loop stability can also be achieved even if the nonlinear system is only Lyapunov stable ($\alpha_e = 0$) for a fixed $\hat{u}_k$. This is a clear advantage of the proposed ESC over classical perturbation based discrete-time ESC techniques that require local asymptotic stability of the nonlinear system. The problem of feedback stabilization of nonlinear discrete-time systems using ESC will be considered in future work.

5 Simulation

In this section, we consider the application of the PIESC approach to nonlinear discrete-time control systems. The performance of the proposed approach is compared to the standard perturbation based ESC algorithm proposed in [7]. This algorithm is given by:

$$\begin{aligned} \xi_{k+1} &= -h_\ell \xi_k + y_k \\ \hat{u}_{k+1} &= \hat{u}_k - \gamma\alpha\cos(\omega k)(y_k - (1+h_\ell)\xi_{k+1}) \\ u_k &= \hat{u}_k + \alpha\cos(\omega(k+1)). \end{aligned}$$

5.1 Example 1

We first consider the application of the PI-ESC approach to the following nonlinear discrete-time system:

$$\begin{aligned} x_{k+1} =& 0.99x_k + (u_k - 0.1)(1 + \frac{1}{2}\sin(x_k)) \\ y_k =& 1 + 0.2(x_k - 1)^2 \end{aligned}$$

We first note that the nonlinear system has a pole very close to the unit circle. The optimum occurs at $x^* = 1$, $u^* = 0.1069$. The PIESC is used with a gain of $k_g = 10$ and integral time constant $\tau_I = 100$. The dither signal is $d_k = 0.05\sin(k)$. The estimation gates are set to $K = 0.001$, $\alpha = 0.001$ and $\sigma = 0.001$. The simulation results are shown in Figure 3. The figure shows the cost function, y_k, the input, u_k, and the integration variable $\hat{u}_k$. The PIESC very effectively converges to the optimum equilibrium conditions. The tuning parameters for the perturbation ESC are $h_\ell = 0.1$, $\gamma = 3/\alpha$, $\alpha = 0.1$, $\omega = 2$. The corresponding ESC performance is shown as the dashed line in Figure 3. As expected, the proposed PIESC provides a drastically faster convergence to the optimum conditions. Furthermore, the impact of the slow nearly unstable dynamics are compensated by the presence of the proportional action.

Next, we consider the following unstable nonlinear system:

$$\begin{aligned} x_{k+1} =& 1.01x_k + (u_k - 0.1)(1 + \frac{1}{2}\sin(x_k)) \\ y_k =& 1 + 0.2(x_k - 1)^2 \end{aligned}$$

The optimum occurs at $x^* = 1$, $u^* = 0.09457$. The PIESC is used with a gain of $k_g = 0.2$ and integral time constant $\tau_I = 1000$. The dither signal is $d_k = 0.5\sin(15k)$. The estimation gains are set to $K = 0.001$, $\alpha = 0.001$ and $\sigma = 0.001$. Figure 4 shows the simulation results. The PIESC simultaneously stabilizes the nonlinear system and identifies the optimum equilibrium conditions. The standard perturbation based ESC technique cannot successfully optimize this system.

To verify the robustness of the proposed approach, random zero mean measurement noise is added to the cost measurement. The noise measurement is given by:

$$y_k = 1 + 0.2(x_k - 1)^2 + 0.03\nu_k$$

where ν_k is a zero mean, unit variance Gaussian random variable. The results are shown in Figure 5. Six simulations are performed. All simulation show good transient performance to the unknown optimum.

5.2 Example 2

The task is to stabilize the nonlinear discrete-time control system studied in [10] given by:

$$\begin{aligned} x_{1,k+1} &= x_{3,k}^2 + u_{1,k} \\ x_{2,k+1} &= x_{2,k} + u_{2,k} \\ x_{3,k+1} &= 2x_{3,k}(u_{1,k} + x_{1,k}x_{2,k}u_{2,k}) \end{aligned}$$

with cost function $y_k = \frac{1}{2}(x_{1,k}^2 + x_{2,k}^2 + x_{3,k}^2)$.

The optimum occurs at $x^* = [0,\, 0,\, 0]^T$, $u^* = [0,\, 0]^T$. The PIESC is used with a gain of $k_g = 0.5$ and integral time constant $\tau_I = 50$. The dither signal is $d_k = [0.2\sin(450k),\, 0.2\sin(400k)]^T$. The estimation gates are set to $K = 0.001$, $\alpha = 0.01$ and $\sigma = 0.01$. Figure 6 shows the output function along with the two inputs. The corresponding state trajectories are shown in Figure 7. The PIESC simultaneously stabilizes the nonlinear system and identifies the optimum equilibrium conditions.

6 Conclusion

This paper proposes a proportional-integral extremum-seeking control technique for a class of discrete-time nonlinear dynamical systems with unknown dynamics. The main contribution of this technique is the minimization of the impact of time-scale separation on the transient performance of the extremum-seeking control system in discrete-time.

References

[1] K. Ariyur and M. Krstic. Analysis and design of multivariable extremum seeking. In *Proceedings of the American Control Conference*, pages 2903–2908, Anchorage, 2002.

[2] K. B. Ariyur and M. Krstic. *Real-time optimization by extremum-seeking control*. Wiley-Interscience, 2003.

[3] J.-Y. Choi, M. Krstic, K. Ariyur, and J. Lee. Extremum seeking control for discrete-time systems. *IEEE Trans. Autom. Contr.*, 47(2):318–323, 2002.

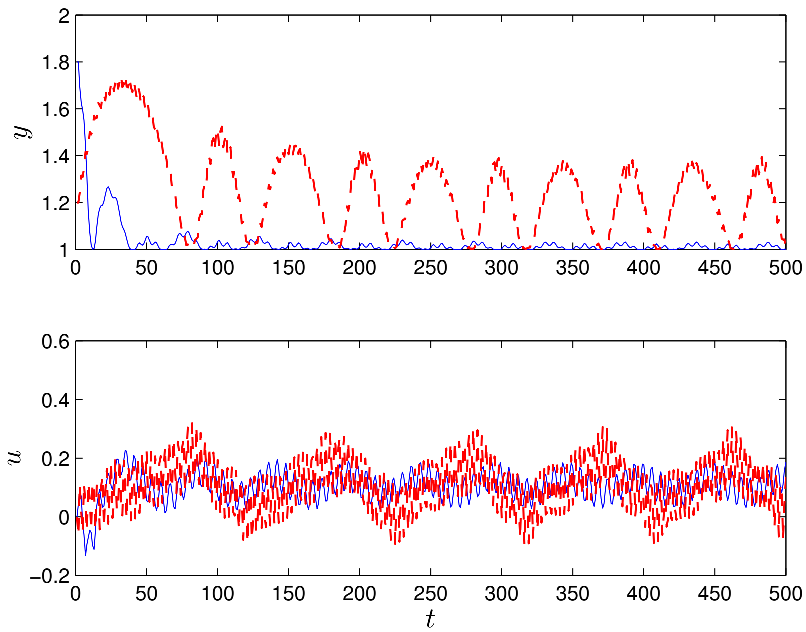

Figure 3: Performance of the PI-ESC for Example 1. The PIESC algorithm trajectories are shown as full lines and the standard ESC, as dashed lines. The PIESC trajectories are shown as full lines and the ESC, as dashed lines.

[4] G. Goodwin and K. Sin. *Adaptive Filtering Prediction and Control*. Dover Publications, Incorporated, 2013.

[5] M. Guay. A time-varying extremum-seeking control approach for discrete-time systems. *Journal of Process Control*, 24(3):98–112, 2014.

[6] M. Guay and D. Dochain. A proportional-integral extremum-seeking controller design technique. In *Proc. IFAC World Congress, Cape Town, South Africa*, 2014.

[7] N. J. Killingsworth and M. Krstic. Pid tuning using extremum seeking: online, model-free performance optimization. *Control Systems, IEEE*, 26(1):70–79, 2006.

[8] M. Krstic. Performance improvement and limitation in extremum seeking control. *Systems and Control Letters*, 39(5):313–326, 2000.

[9] M. Krstic and H. Wang. Stability of extremum seeking feedback for general dynamic systems. *Automatica*, 36(4):595–601, 2000.

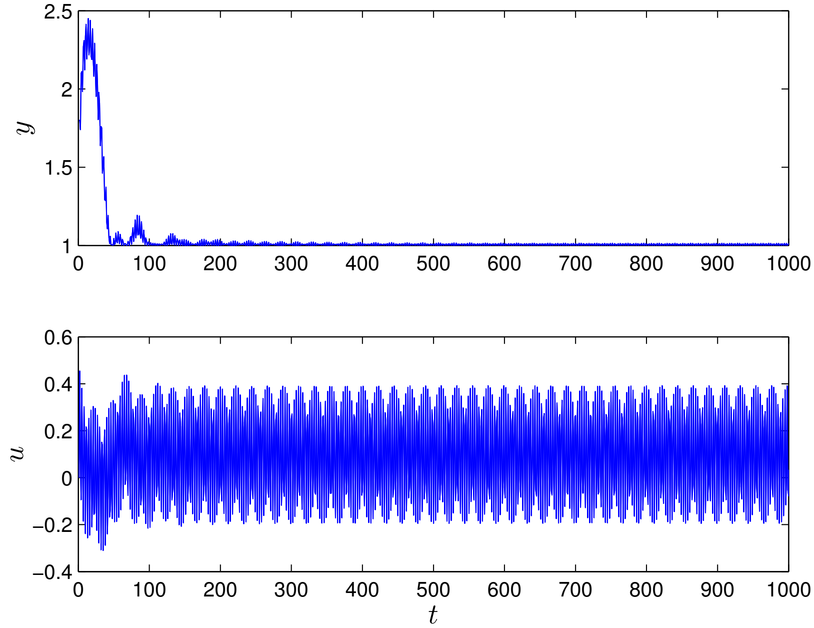

Figure 4: Performance of the PI-ESC for the unstable discrete-time system in Example 1. The upper plot shows the cost function and the bottom plot shows the input variable as a function of the sampling time k.

[10] W. Lin. Further results on global stabilization of discrete nonlinear systems. *Systems & Control Letters*, 29(1):51–59, 1996.

[11] S.-J. Liu and M. Krstic. Discrete-time stochastic extremum seeking. In *Proc. IFAC World Congress*, volume 19, pages 3274–3279, 2014.

[12] C. Manzie and M. Krstic. Extremum seeking with stochastic perturbations. *IEEE Trans. Autom. Contr.*, 54(3):580–585, 2009.

[13] D. Nesic, T. Nguyen, Y. Tan, and C. Manzie. A non-gradient approach to global extremum seeking: An adaptation of the shubert algorithm. *Automatica*, 49(3):809 – 815, 2013.

[14] J. J. Ryan and J. L. Speyer. Peak-seeking control using gradient and hessian estimates. In *American Control Conference (ACC), 2010*, pages 611–616. IEEE, 2010.

[15] B. Srinivasan. Real-time optimization of dynamic systems using multiple units. *International Journal of Robust and Nonlinear Control*, 17(13):1183–1193, 2007.

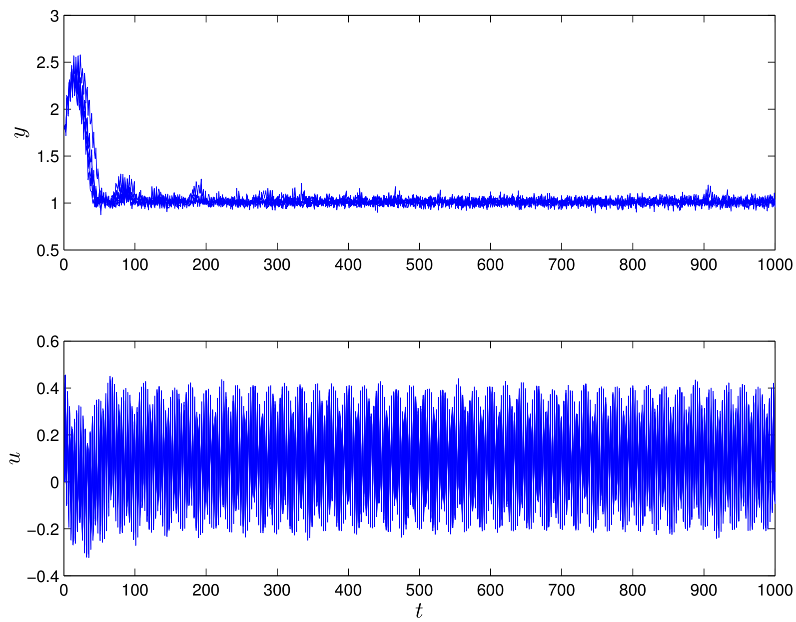

Figure 5: Performance of the PI-ESC for the discrete-time system in Example 1. The upper plot shows the cost function and the bottom plot shows the input variable as a function of the sampling time k for six simulations subject random measurement noise is added.

[16] Y. Tan, W. Moase, C. Manzie, D. Nesic and, and I. Mareels. Extremum seeking from 1922 to 2010. In *29th Chinese Control Conference (CCC)*, pages 14 –26, july 2010.

[17] A. Teel and D. Popovic. Solving smooth and nonsmooth multivariable extremum seeking problems by the methods of nonlinear programming. In *Proceedings of the 2001 American Control Conference*, volume 3, pages 2394–2399. IEEE, 2001.

[18] H. Wang, M. Krstic, and G. Bastin. Optimizing bioreactors by extremum seeking. *International Journal of Adaptive Control and Signal Processing*, 13:651–669, 1999.

[19] C. Zhang and R. Ordóñez. Robust and adaptive design of numerical optimization-based extremum seeking control. *Automatica*, 45(3):634–646, 2009.

[20] C. Zhang and R. Ordóñez. *Extremum-Seeking Control and Applications.* Springer, 2012.

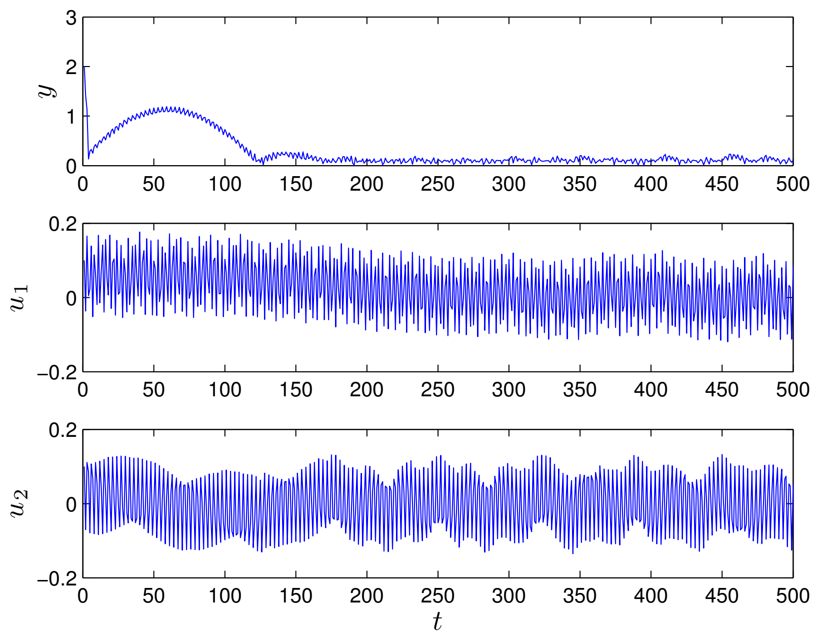

Figure 6: Performance of the PI-ESC for Example 2. The upper plot shows the cost function, the middle plot, the input variable u_1 and the bottom plot, the input variable u_2 as a function of the sampling time k.

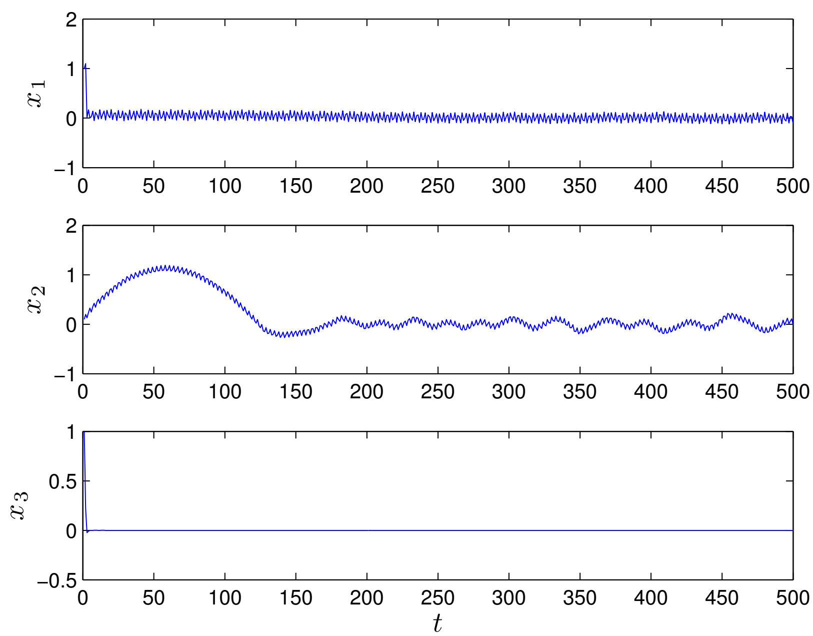

Figure 7: Performance of the PI-ESC for Example 2. The upper plot shows the state variable x_1, the middle plot, x_2 and the bottom plot, x_3 as a function of the sampling time k.