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Robust Soft-Landing Control with Quantized Input

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Abstract: We propose a controller architecture for soft-landing control with quantized input. The objective of the soft-landing problem is to achieve precise positioning of a moving object at a target position, while ensuring the velocity decreases as the target is approached. In this paper, we formulate the soft-landing problem as a constrained control problem. Our approach combines traditional convex model predictive control with a rounding rule that quantizes the input. The rounding rule is designed to minimize the error between the requested and quantized inputs. A robust control invariant set is used to ensure that the rounding errors do not lead to constraint violations. We demonstrate our approach for a transportation system case study.

Keywords: Robust Model Predictive Control, Invariant Sets, Quantized Control

1. INTRODUCTION

Many control applications in the automotive, aerospace, manufacturing, and transportation fields require the precise positioning of a moving object at a desired location while ensuring the the velocity of the object decreases as the target is approached. The resulting soft-landing (also called soft-contact) avoids damage and reduces wear. One example of soft-landing is the control of valves in camless engines, which requires the high-speed closing of a valve in its seating while avoiding rough impacts that reduce component operating life (see Hoffmann et al. (2003)). Another example is docking of spacecraft which requires the docking spacecraft to make soft-contact to avoid the spacecraft ricocheting or suffering damage (see Weiss et al. (2012)). Soft-landing is also important for rider comfort during the automatic stopping of vehicles (see Bu and Tan (2007)).

The soft-landing problem can be formulated as a constrained control problem where constraints are placed on the object velocity relative to its position so that the object slows as it approaches the target position. Hence the soft-landing problem can be solved using constrained control techniques. An early approach to the soft-landing problem was based on reference governors (see Kolmanovsky and Gilbert (2001)). This approach can guarantee constraint satisfaction but has limited performance since reference governors can only manipulate the reference of a linearly pre-compensated system. More recently, model predictive control (MPC) has been used to solve the soft-landing problem (see Di Cairano et al. (2014)). In Di Cairano et al. (2007) soft-landing MPC was applied to soft-landing for valves in camless engines. In Di Cairano et al. (2012) and Weiss et al. (2012) soft-landing MPC was applied to soft-landing for space-craft docking.

In this paper, we consider the soft-landing problem when the input is restricted to a finite set. Model predictive control has been applied to system with finite input in the literature. Aguilera and Quevedo (2011) studied the stabilization of systems with a finite number of inputs using model predictive control. Corona et al. (2006) focused on the optimality of model predictive control for finite input system. In the soft-landing problem, our main concern is with guaranteeing constraint satisfaction rather than optimality or stability. Picasso et al. (2002) presented a method for computing control invariant sets for linear systems with finite input. For numerical simplicity, we adopt the rounding rule approach from Kirches (2011). Our approach combines a convex model predictive controller which guarantees robust state constraint satisfaction and a rounding rule that ensures the input lies in the finite set. Our rounding rule is designed using Voronoi partitions. This approach has been previously used in Bullo and Liberzon (2006). The convex MPC and rounding rule are designed jointly to ensure robust constraint satisfaction.

This paper is organized as follows. In Section 2 we formally define the dynamics, constraints, and control objectives for the soft-landing problem. In Section 3 we describe our control algorithm for solving the soft-landing problem. We pay particular attention to the issue of ensuring our controller is robust to model uncertainty and rounding errors. In Section 4 we demonstrate our control algorithm on a transportation system case study.

2. SOFT-LANDING PROBLEM

In this section we define the dynamics, constraints, and control objectives of the soft-landing problem.

2.1 Soft-Landing Dynamics

We consider an inertial object moving in a one-dimensional space described by the dynamics

$$\dot{x}_m(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix}}_{\bar{A}_m} x_m(t) + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{\bar{B}_m} q_f(t) \tag{1}$$

where the state $x_m(t) = [y(t), \dot{y}(t)]^T \in \mathbb{R}^2$ of the systems is the position $y(t)$ and velocity $\dot{y}(t)$ of the inertial object, m is the mass of the object, b is the viscous friction coefficient, and $q_f(t) \in \mathbb{R}^1$ is the controlled force on the object.

The controlled force $q_f(t)$ is the output of a linear filter that captures the dynamics of the actuator. The filter is described by the state-space model

$$\dot{x}_q(t) = \bar{A}_q x_q(t) + \bar{B}_q q(t) \tag{2a}$$

$$q_f(t) = \bar{C}_q x_q(t) + \bar{D}_q q(t) \tag{2b}$$

where $x_q \in \mathbb{R}^{n_q}$ is the state of the input filter and $q(t) \in \mathbb{R}^m$ is the input command. The soft-landing problem assumes specific dynamics and constraints for the inertial subsystem, but the input dynamics can be arbitrary and are unconstrained. The soft-landing problem can also include a disturbance force d. In addition the controlled input force $q_f(t)$ can depend on the state $x_m(t)$ of the inertia systems. However, for simplicity, we do not consider these cases in this paper.

The state of the composite system $x(t) = [x_m(t), x_q(t)]$ is the state of the inertial system $x_m(t)$ and input filter $x_q(t)$. In discrete-time, the dynamics of the composite system are modeled by

$$\begin{bmatrix} x_m(k+1) \\ x_q(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{A}_m & \hat{B}_m \hat{C}_q \\ 0 & \hat{A}_q \end{bmatrix}}_{A} \begin{bmatrix} x_m(k) \\ x_q(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \hat{B}_m \hat{D}_q \\ \hat{B}_q \end{bmatrix}}_{B} q(k) \tag{3}$$

where $(\hat{A}_m, \hat{B}_m)$ and $(\hat{A}_q, \hat{B}_q, \hat{C}_q, \hat{D}_q)$ are the discrete-time transformations of $(\bar{A}_m, \bar{B}_m)$ and $(\bar{A}_q, \bar{B}_q, \bar{C}_q, \bar{D}_q)$ respectively. We use the short-hand $x(k) = x(t_k)$ and $q(k) = q(t_k)$ to denote the state and input, respectively, at time $t_k = t_0 + k\Delta t$ where $k \in \mathbb{N}$ and Δt is the discrete-time sample period.

2.2 Soft-Landing Constraints

In this section we describe the state and input constraints for the soft-landing problem. The state constraints only apply to the state $x_m(t)$ of the inertial system.

The objective of the soft-landing problem is to bring the inertial object to a stop in a neighborhood of the origin, called the target set

$$\mathcal{T} = \left\{ \begin{bmatrix} x_{m,1} \\ x_{m,2} \end{bmatrix} : \begin{array}{r} x_{min} \leq x_{m,1} \leq x_{max} \\ x_{m,2} = 0 \end{array} \right\} \subseteq \mathbb{R}^2$$

where $x_{min} < 0 < x_{max}$ are the maximum deviations of the position $x_{m,1}$ from the origin. We only require that the state $x_m(t)$ reaches the target set, not that it remains in the target set. This is motivated by several practical applications in which, upon entering the target set, the dynamics change in such a way to keep the state in the target set. For instance static friction is used to hold valves in their seating, clamps are used hold spacecraft in place during docking, and parking brakes are used to keep elevators at floor level.

We do not want the state $x_m(k)$ of the inertial systems to approach the target set $\mathcal{T}$ with a velocity that is too fast or too slow. Therefore we introduce the "soft-landing" cone constraint to control the approach velocity

$$\mathcal{S} = \left\{ \begin{bmatrix} x_{m,1} \\ x_{m,2} \end{bmatrix} : \begin{array}{r} x_{m,2} + \gamma_{max}(x_{m,1} - x_{max}) \leq 0 \\ x_{m,2} + \gamma_{min}(x_{m,1} - x_{min}) \geq 0 \end{array} \right\} \tag{4}$$

where $\gamma_{max}, \gamma_{min} \in \mathbb{R}_+$ with $\gamma_{min} < \gamma_{max}$ are spatial deceleration coefficients. This constraint bounds the velocity $x_{m,2}(k) = \dot{y}(t_k)$ of the inertial system as a function of position $x_{m,1}(k) = y(t_k)$ to ensures the velocity decreases smoothly as the inertial system approaches the target set $\mathcal{T} \subseteq \mathbb{R}^2$. The state constraint set is given by the unbounded polytope

$$\mathcal{X} = \mathcal{S} \times \mathbb{R}^{n_q} \subset \mathbb{R}^n \tag{5}$$

where $n = 2 + n_q$ is the dimension of the composite system (3). In the soft-landing problem there are no constraints on the state of the input $x_q \in \mathbb{R}^{n_q}$ filter.

The nonlinearity of this problem is due to the quantization of the input $q(k)$ which is drawn from a finite set $\mathcal{Q} \subset \mathbb{R}^m$ where $|\mathcal{Q}| < \infty$. We assume that the convex-hull $\text{conv}(\mathcal{Q})$ of the input set $\mathcal{Q}$ contains the origin in its interior $0 \in \text{conv}(\mathcal{Q})$.

2.3 Soft-Landing Objectives

The objective of the soft-landing problem is to generate an input trajectory that drives the inertial object to the target set while satisfying state constraints. The soft-landing problem is formally stated below.

Problem 1. (Soft-landing). Select a feasible input trajectory $q(k) \in \mathcal{Q}$ for $k \in \mathbb{N}$ such that the state trajectory $x(k) = [x_m(k), x_q(k)]^T$ resulting from the dynamics (3) satisfies the state constraints $x(k) \in \mathcal{X}$ for all $k \in \mathbb{N}$ and converges $x(k) \to \mathcal{T}$ to the target set $\mathcal{T}$ i.e. there exists $f \in \mathbb{N}$ such that $x(f) \in \mathcal{T}$.

If the final time $f = \infty$ is infinite, then we mean that the state asymptotically converges the target set $x_m(k) \to \mathcal{T}$.

In Di Cairano et al. (2014) it was shown that a feasible state trajectory $x(k) \in \mathcal{X}$ that satisfies the dynamics (3) will necessarily converge to the target set. Thus Problem 1 can be solved by simply finding a feasible input trajectory $q(k) \in \mathcal{Q}$ that produces a persistently feasible state trajectory $x(k) \in \mathcal{X}$. In Di Cairano et al. (2014), Problem 1 was solved using convex model predictive control with a robust control invariant set that guaranteed persistent feasibility. In this paper we extend this result to the case where the input is quantized.

3. SOFT-LANDING CONTROL DESIGN

In this section we describe our controller for solving the Soft-Landing Problem 1. Our controller consists of two parts connected in series: a convex model predictive controller and a rounding rule. The convex model predictive controller is used to ensure robust state constraint satisfaction. The rounding rule is used to ensure satisfaction of the input constraint. The rounding rule will be described in Section 3.2 and the model predictive controller will be described in Section 3.3.

3.1 Model Uncertainty

The mass m and the viscous friction coefficient b of the inertial system are uncertain and thus the matrices $\bar{A}_m$ and $\bar{B}_m$ are uncertain. In addition, the dynamics matrices $(\bar{A}_q, \bar{B}_q, \bar{C}_q, \bar{D}_q)$ of the input dynamics (2) may be uncertain. Therefore the dynamics matrices A and B of the composite system (3) are uncertain.

The parametric uncertainty of the composite system (3) is modeled by the polytopic linear parameter varying system

$$x(k+1) = A(\xi)x(k) + B(\xi)q(k) \tag{6}$$

where A_i and B_i for $i \in \mathcal{I}$ are a finite collection $|\mathcal{I}| < \infty$ of extreme dynamics whose convex-hulls $A(\xi) = \sum_{i\in\mathcal{I}} \xi_i A_i$ and $B(\xi) = \sum_{i\in\mathcal{I}} \xi_i B_i$ cover all possible realizations of the actual system dynamics (A, B), and $\xi \in \Xi = \{\xi : \xi \geq 0, \sum_{i\in\mathcal{I}} \xi_i = 1\}$ is an unknown time-varying parameter vector.

3.2 Rounding Rule

In this section we describe the rounding rule we use to ensure the control input $q(k)$ lies inside the finite set $\mathcal{Q}$.

The convex model predictive controller will relax the non-convex constraint $q \in \mathcal{Q}$. It will compute control inputs $u \in \mathcal{U}$ in a set $\mathcal{U} \supseteq \text{conv}(\mathcal{Q})$ containing the convex-hull $\text{conv}(\mathcal{Q})$ of the finite-set $\mathcal{Q}$. A rounding rule $\mathbf{q} : \mathcal{U} \to \mathcal{Q}$ is used to map the requested input $u \in \mathcal{U}$ to a feasible quantized input $q = \mathbf{q}(u) \in \mathcal{Q}$. We propose a rounding rule of the form

$$\mathbf{q}(u) = \underset{q\in\mathcal{Q}}{\text{argmin}}\, \|u - q\|_W^2 \tag{7}$$

where the weighting matrix $W \in \mathbb{R}^{m\times m}$ is a design parameter use to ameliorate the effects of rounding the requested input $u \in \mathcal{U}$. The offline design of the weighting matrix W will be discussed later in this section.

The difference between the requested control input $u \in \mathcal{U}$ and the implemented control input $\mathbf{q}(u) \in \mathcal{Q}$ is treated as a disturbance on the system

$$w = u - \mathbf{q}(u). \tag{8}$$

We will bound the set of possible rounding errors (8) using Voronoi cells. The Voronoi cell of a point $q \in \mathcal{Q}$ in a finite set $\mathcal{Q}$ is the set of points $u \in \mathcal{U} \supseteq \text{conv}(Q)$ closer to $q \in \mathcal{Q}$ than any other point $p \in \mathcal{Q}$ in the set $\mathcal{Q}$

$$\mathcal{V}(q, \mathcal{Q}) = \big\{u \in \mathcal{U} : \|u - q\|_W^2 \leq \|u - p\|_W^2,\ \forall p \in \mathcal{Q}\big\}.$$

The Voronoi cell $\mathcal{V}(q, \mathcal{Q}) \subseteq \mathbb{R}^m$ is a polytope. If the rounding rule (7) chooses the quantized input $q \in \mathcal{Q}$ for $u \in \mathcal{U}$, then the rounding error (8) must lie in the shifted Voronoi cell $\mathcal{V}(q, \mathcal{Q}) - q \subseteq \mathbb{R}^m$. Thus the set of all possible rounding errors is the union $\bigcup_{q\in\mathcal{Q}} \mathcal{V}(q, \mathcal{Q}) - q \subseteq \mathbb{R}^m$ of shifted Voronoi cells $\mathcal{V}(q, \mathcal{Q}) - q$. This set is generally non-convex, hence we will outer-bound it by its convex-hull

$$\mathcal{W} = \text{conv}\big\{\mathcal{V}(q, \mathcal{Q}) - q : q \in \mathcal{Q}\big\}. \tag{9}$$

We can conservatively assume that all rounding errors (8) produced by the rounding rule (7) will lie in the polytopic set (9). In the next section, we will use standard robust model predictive control techniques to design a controller that is robust to rounding errors $w \in \mathcal{W}$ in the set (9). Thus we can guarantee constraint satisfaction despite the quantization of the input.

Next we discuss how we choose the weighting matrix $W \in \mathbb{R}^{m\times m}$ to shape the set (9) of possible rounding errors. The rounding rule (7) chooses the quantized input $\mathbf{q}(u)$ that minimizes the quantity $w^T W w$ where $w = u - \mathbf{q}(u)$. Thus we would like to choose the matrix $W \in \mathbb{R}^{m\times m}$ so that if the quantity $w^T W w$ is small, then the effect of the rounding error $w = u - \mathbf{q}(u)$ on the system is small.

For a given polytopic input set $\mathcal{U} \supseteq \text{conv}(\mathcal{Q})$, we compute the ideal weighting matrix W offline by solving a series of semi-definite programs

$$\text{maximize}\ \log\det W^{-1} \tag{10a}$$

$$\text{subject to}\ \begin{bmatrix} (1-\alpha)P^{-1} & * & * \\ A_i P^{-1} + B_i F P^{-1} & P^{-1} & * \\ 0 & W^{-1}B_i^T & \alpha W^{-1} \end{bmatrix} \succeq 0 \tag{10b}$$

$$\begin{bmatrix} P^{-1} & * \\ H_{x,i}^T & K_{x,i}^2 \end{bmatrix} \succeq 0 \text{ for } i = 1, \ldots, c_{\mathcal{X}} \tag{10c}$$

$$\begin{bmatrix} P^{-1} & * \\ H_{u,i}^T F P^{-1} & K_{u,i}^2 \end{bmatrix} \succeq 0 \text{ for } i = 1, \ldots, c_{\mathcal{U}} \tag{10d}$$

where $(H_{x,i}, K_{x,i})$, and $(H_{u,i}, K_{u,i})$ are the half-space parameters that define the state and relaxed input constraint sets respectively

$$\mathcal{X} = \big\{x \in \mathbb{R}^n : H_{x,i}^T x \leq K_{x,i} \text{ for } i = 1, \ldots, c_{\mathcal{X}}\big\}$$
$$\mathcal{U} = \big\{u \in \mathbb{R}^m : H_{u,i}^T x \leq K_{u,i} \text{ for } i = 1, \ldots, c_{\mathcal{U}}\big\}.$$

Problem (10) finds the largest ellipsoidal set of rounding errors

$$\mathcal{E}(W^\star) = \big\{w \in \mathbb{R}^m : w^T W^\star w \leq 1\big\} \subseteq \mathbb{R}^m$$

such that the uncertain system (6), for all realizations of the dynamics $\xi \in \Xi$, has an ellipsoidal positive invariant set

$$\mathcal{E}(P^\star) = \big\{x \in \mathbb{R}^n : x^T P^\star x \leq 1\big\} \subseteq \mathbb{R}^n$$

under a linear controller of the form $u = F^\star x$ that satisfies the state constraints $\mathcal{E}(P^\star) \subseteq \mathcal{X}$ and the relaxed input constraints $F^\star\mathcal{E}(P^\star) \subseteq \mathcal{U}$. This is proven in Theorem 1 below. Since $\mathcal{E}(W^\star)$ is the largest set of rounding errors that the uncertain system (6) can reject using linear control without violating state or input constraints, the effects of the errors in the set $\mathcal{E}(W^\star)$ must be relatively small. We can formalize this intuition using the Minkowski function of the set $\mathcal{E}(W^\star)$ to measure the size of the rounding error w

$$\Phi_{\mathcal{E}(W^\star)}(w) = \min\big\{\lambda \geq 0 : w \in \lambda\mathcal{E}(W^\star)\big\} = \|w\|_{W^\star}^2$$

The rounding rule (7) chooses the quantized input $q \in \mathcal{Q}$ that minimizes the rounding error in terms of the Minkowski function of the set $\mathcal{E}(W^\star)$.

Theorem 1. Let $W^\star \in \mathbb{R}^{m\times m}$, $P^\star \in \mathbb{R}^{n\times n}$, and $F^\star \in \mathbb{R}^{m\times n}$ be the optimal solution of problem (10). Then $\mathcal{E}(W^\star)$ is the largest ellipsoidal disturbance set such that the system (6) for all realizations of the dynamics $\xi \in \Xi$ has a positive invariant set $\mathcal{E}(P^\star)$ under a linear controller $u = F^\star x$ that satisfies state $\mathcal{E}(P^\star) \subseteq \mathcal{X}$ and input $F^\star\mathcal{E}(P^\star) \subseteq \text{conv}(\mathcal{U})$ constraints.

An immediate corollary of Theorem 1 is that if the rounding error set (9) is contained $\mathcal{W} \subseteq \mathcal{E}(W^\star)$ in the ellipsoid $\mathcal{E}(W^\star)$ then the set $\mathcal{E}(P^\star)$ is positive invariant under model uncertainty and rounding errors.

Corollary 2. Suppose the rounding error set (9) is contained $\mathcal{W} \subseteq \mathcal{E}(W^\star)$ in the ellipsoid $\mathcal{E}(W^\star)$ where $W^\star$, $P^\star$, and $F^\star$ is the solution to problem (10). Then the set $\mathcal{E}(P^\star)$ is positive invariant for the system (6) in closed-loop with the linear controller $u = F^\star x$.

Corollary 2 means that it is possible to satisfy constraints despite model uncertainty and rounding errors. Thus it is possible to solve the soft-landing problem with an uncertain model and a finite set of inputs $\mathcal{Q}$.

The matrix inequalities in problem (10) are linear in the decision variables $X = P^{-1}$, $Y = W^{-1}$, and $Z = FP^{-1}$. Thus, for a fixed $\alpha \in [0, 1]$, problem (10) is a semi-definite program. Therefore we can find the the solution $W^\star \succeq 0$ by performing a line-search on the scalar parameter α and solving a series of convex semi-definite programs.

The solution of problem (10) depends on the choice of the relaxed input set $\mathcal{U} \supseteq \text{conv}(\mathcal{Q})$. We would like the input set $\mathcal{U}$ to be large to produce a less conservative positive invariant set $\mathcal{E}(P^\star)$. However a larger input set $\mathcal{U}$ will produce a larger set of rounding errors $\mathcal{W}$. We use Algorithm 1 as a heuristic for determining the relaxed input set $\mathcal{U}$. Algorithm 1 initially choses the relaxed input set $\mathcal{U}_0 = \text{conv}(\mathcal{Q})$. During each iteration, Algorithm 1 finds the corresponding error set $\mathcal{W}_k$ by solving problem (10) and using the weighting matrix $W_k^\star$ to determine the rounding error set (9). We note that the non-convex input set $\hat{\mathcal{U}}_{k+1} = \mathcal{Q} \oplus \mathcal{W}_k$ produces the same errors set (9) as the input set $\mathcal{U}_k$. Algorithm 1 updates the input set $\mathcal{U}_{k+1} = \text{conv}(\mathcal{Q} \oplus \mathcal{W}_k)$ by taking the convex relaxation of the set $\mathcal{Q} \oplus \mathcal{W}_k$. Algorithm 1 terminates when the rounding error set $\mathcal{W}_k$ is too large $\mathcal{W}_k \not\subseteq \mathcal{E}(W^\star)$ for a linear controller $u = F_k^\star x$ to reject without violating state or input constraints.

Algorithm 1 Computation of relaxed input set $\mathcal{U}$

$\mathcal{U}_0 = \text{conv}(\mathcal{Q})$
repeat
  Compute $W_k^\star$ by solving problem (10) for $\mathcal{U} = \mathcal{U}_k$.
  Compute the error set $\mathcal{W}_k$ using (9)
  Update input set $\mathcal{U}_{k+1} = \text{conv}(\mathcal{Q} \oplus \mathcal{W}_k)$
until $\mathcal{W}_k \not\subseteq \mathcal{E}(W_k^\star)$
$\mathcal{U} = \mathcal{U}_k$

3.3 Soft-landing Model Predictive Controller

In this section we describe the model predictive controller used for the soft-landing problem.

During each sample period, the controller executes the following steps online:

(1) Measure (or estimate) the current state $x(k) = [x_m(k), x_q(k)]^T$ of the composite system (3).
(2) Solve the following finite time optimal control problem

$$\underset{v_0,\dots,v_{N-1}}{\text{minimize}} \quad x_N^T P x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k \tag{11a}$$

$$\text{subject to} \quad x_{k+1} = A(\hat{\xi}) x_k + B(\hat{\xi}) u_k \tag{11b}$$

$$x_k \in \mathcal{X}, u_k \in \mathcal{U} \tag{11c}$$

$$(x_0, u_0) \in \mathcal{C}_{x,u} \tag{11d}$$

where x_k is the predicted state trajectory starting from the initial state $x_0 = x(k)$ under the control input u_k over the horizon N. The nominal dynamics (11b) use an estimate $\hat{\xi} \in \Xi$ of the unknown parameter $\xi \in \Xi$ which is constant over the prediction horizon.
(3) Apply the rounding rule (7) to obtain a feasible input $\mathbf{q}(u_0^\star) \in \mathcal{Q}$.

Next we discuss the offline design of the constraint set $\mathcal{C}_{x,u} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ used in the constrained finite-time optimal control problem (11).

To guarantee persistent constraint satisfaction, the model predictive controller will restrict the state $x = [x_m, x_q]^T$ of the uncertain system (6) to a control invariant subset $\mathcal{C} \subseteq \mathcal{X}$ of the state-space $\mathcal{X}$. The control invariant set $\mathcal{C}$ must be robust to model uncertainty and rounding errors. Robust control invariant sets are defined below.

Definition 3. A set $\mathcal{C} \subseteq \mathcal{X}$ is a robust control invariant set if for all $x \in \mathcal{C}$ there exists a control input $q \in \mathcal{Q}$ such that $A(\xi)x + B(\xi)q \in \mathcal{C}$ for all realizations of the dynamics $\xi \in \Xi$. A set $\mathcal{C}^\infty \subseteq \mathcal{X}$ is the maximal robust control invariant set if it contains $\mathcal{C}^\infty \supseteq \mathcal{C}$ all other robust control invariant sets $\mathcal{C} \subseteq \mathcal{X}$.

Computation of a robust control invariant set uses the predecessor operator

$$\text{Pre}(\Omega) = \big\{x \in \mathbb{R}^n : \exists u \in \mathcal{U} \text{ s.t. } A_i x + B_i u - B_i w \in \Omega, \\ \text{for all } i \in \mathcal{I} \text{ and } w \in \mathcal{W}\big\}. \tag{12}$$

The predecessor operator (12) computes the set of states $x \in \text{Pre}(\Omega)$ that can be mapped into the set Ω for all rounding errors $w \in \mathcal{W}$ and extreme dynamics (A_i, B_i) for $i \in \mathcal{I}$.

Algorithm 2 Computation of robust control invariant set

$\Omega_0 = \mathcal{X}$
repeat
  $\Omega_{k+1} = \text{Pre}(\Omega_k) \cap \Omega_k$
until $\Omega_{k+1} = \Omega_k$
$\mathcal{C} = \Omega_k$

A robust control invariant set is computed using Algorithm 2. We refer the reader to Blanchini and Miani (2007) for details about Algorithm 2. The following proposition shows that the set $\mathcal{C}$ produced by Algorithm 2 is a robust control invariant set for the system (6) with quantized input $q \in \mathcal{Q}$.

Proposition 4. The set $\mathcal{C} \subseteq \mathcal{X}$ produced by Algorithm 2 is a robust control invariant set for the uncertain system (6) subject to state constraints $x \in \mathcal{X}$ and quantized input constraints $q \in \mathcal{Q}$.

Remark 1. The set $\mathcal{C}$ produced by Algorithm 2 is a robust control invariant set, but not necessarily the maximal robust control invariant set since the rounding error set (9) is a conservative over-approximation of the actual non-convex set of rounding errors. The maximal robust control invariant set for the uncertain system (6) with finite input $q \in \mathcal{Q}$ will, in general, be non-convex. Finding the maximal robust control invariant set for this problem is computationally difficult since it requires computing an exponential union of polytopes. In addition for a non-convex control invariant set, the constrained finite-time optimal control problem (11) is difficult to solve online since it is a non-convex problem. Instead, we compute a convex inner-approximation of the maximal robust control invariant set using Algorithm 2. A convex inner-approximation is easier to compute offline and makes the resulting constrained finite-time optimal control problem easier to solve online.

We define the admissible set $\mathcal{C}_{x,u} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ as the set of state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$ pairs such that the successive state $A(\xi)x+B(\xi)u-B(\xi)w$ is inside the control invariant set $\mathcal{C}$ for all rounding errors $w \in \mathcal{W}$ and all realizations $\xi \in \Xi$ of the dynamics

$$\mathcal{C}_{x,u} = \big\{(x,u) : A(\xi)x+B(\xi)u - B(\xi)w \in \mathcal{C} \quad \text{for all } \xi \in \Xi \text{ and } w \in \mathcal{W}\big\}. \tag{13}$$

By convexity the admissible set (13) is equivalent to the set

$$\mathcal{C}_{x,u} = \big\{(x,u) : A_i x+B_i u - B_i w \in \mathcal{C} \quad \text{for all } i \in \mathcal{I} \text{ and } w \in \mathcal{W}\big\}.$$

The model predictive controller requires (11d) that the initial state x_0 and input u_0 lie in the admissible set $\mathcal{C}_{x,u}$. This ensures persistent feasibility of optimal control problem (11). Since the invariant set $\mathcal{C}$ is robust to model uncertainty and rounding errors, the model predictive controller and rounding rule will satisfy constraints. As shown in Di Cairano et al. (2014), persistent constraint satisfaction implies that the controller solves the soft-landing problem (Problem 1).

Our model predictive controller tacitly assumes that the uncertain system (6) subject to rounding errors (8) has a robust control invariant set. The existence of such a set is guaranteed by the condition $\mathcal{W} \subseteq \mathcal{E}(W^\star)$ in Corollary 2. If this condition is not satisfied then it may not be possible to solve the soft-landing problem with this finite input set $\mathcal{Q}$.

If Algorithm 2 converges after a finite number of iterations, then the robust control invariant set $\mathcal{C} \subseteq \mathcal{X}$ will be a polytope $\mathcal{C} = \{x \in \mathbb{R}^n : H_{\mathcal{C}} x \leq K_{\mathcal{C}}\}$. Likewise the admissible set $\mathcal{C}_{x,u}$ will be polytope

$$\mathcal{C}_{x,u} = \big\{(x,u) : H_{\mathcal{C}} A_i x + H_{\mathcal{C}} B_i u \leq K_{\mathcal{C}}, \forall i \in \mathcal{I}\big\}$$

since the set of extreme systems $\mathcal{I}$ is finite $|\mathcal{I}| < \infty$. Thus the constrained finite-time optimal control problem (11) can be posed as a quadratic program. Our controller can be implemented on modest hardware since it only requires solving a quadratic program and evaluating a simple rounding rule (7) online.

4. CASE-STUDY IN TRANSPORTATION SYSTEMS

In this section we apply our soft-landing controller to the problem of stopping a large automated transportation vehicle moving along a fixed track.

The transportation vehicle has dynamics of the form (1). The mass m and viscous friction coefficient b are uncertain, but lie in the intervals $[\underline{m}, \overline{m}]$ and $[\underline{b}, \overline{b}]$ respectively. The targets set is the set of positions $x_{m,1}$ within 2 meters of the origin

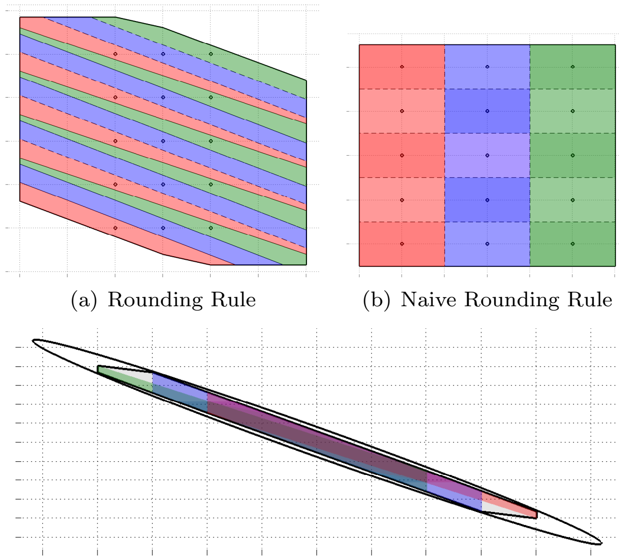
$$\mathcal{T} = \left\{ \begin{bmatrix} x_{m,1} \\ x_{m,2} \end{bmatrix} : \begin{matrix} -2 \leq x_{m,1} \leq 2 \\ x_{m,2} = 0 \end{matrix} \right\}.$$

The vehicle has two actuators: a pneumatic actuator with high control authority but slow dynamics and an electric actuator with fast dynamics but low control authority. The controlled input $q = [q_e, q_a]^T$ is the force from both actuators whose dynamics are modeled by a low-pass filter

$$\dot{x}_q(t) = -\tfrac{1}{\tau_a} x_q(t) + \left[\tfrac{1}{\tau_a},\ 0\right] \begin{bmatrix} q_e(t) \\ q_a(t) \end{bmatrix} \tag{14a}$$

$$q_f(t) = k_a x_q(t) + [0,\ k_e] \begin{bmatrix} q_e(t) \\ q_a(t) \end{bmatrix} \tag{14b}$$

where q_e and q_a are the commands to the electric and pneumatic actuators respectively, τ_a and k_a are, respectively, the time-constant and gain of the pneumatic actuator, and k_e is the gain of the electric actuator. The actuator gains k_a and k_a are uncertain but lie in the intervals $k_a \in [\underline{k}_a, \overline{k}_a]$ and $k_e \in [\underline{k}_e, \overline{k}_e]$ respectively.



(a) Rounding Rule (b) Naive Rounding Rule

(c) Rounding Error Set

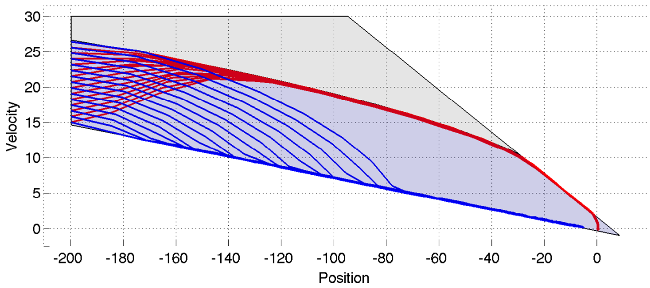
Fig. 1. (a) Rounding rule used by the controller. (b) Naive rounding rule which rounds control input to the nearest discrete value. Circles are the elements of the finite input set $\mathcal{Q}$ and polytopes are the Voronoi cells $\mathcal{V}(q, \mathcal{Q})$ of the inputs $q \in \mathcal{Q}$. (c) Ellipsoidal error set $\mathcal{E}(W)$ enclosing the rounding error sets $\mathcal{V}(q, \mathcal{Q}) - q$ and their convex-hull $\mathcal{W} = \mathrm{conv}\{\mathcal{V}(q, \mathcal{Q}) - q : q \in \mathcal{Q}\}$.

The electric actuator has three settings: full-throttle $q_e = 1$, idle $q_e = 0$, and braking $q_e = -1$. The pneumatic actuator has five settings: full-throttle $q_a = 1$, half-throttle $q_a = 0.5$, idle $q_a = 0$, half-braking $q_a = -0.5$, and full-braking $q_e = -1$. Thus the input $q \in \mathcal{Q}$ is restricted to the $|\mathcal{Q}| = 15$ element set

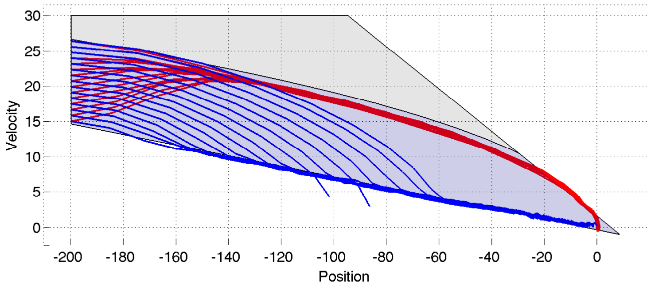
$$\mathcal{Q} = \big\{-1, 0, 1\big\} \times \big\{-1, -0.5, 0, 0.5, 1\big\}.$$

Figure 1(a) shows the rounding rule (7) where relaxed input set $\mathcal{U}$ and weighting matrix $W^\star$ were obtained using Algorithm 1. The circles correspond to the $|\mathcal{Q}| = 15$ elements of the finite input set $\mathcal{Q}$ and the surrounding polytopes are the Voronoi cells $\mathcal{V}(q, \mathcal{Q})$ for each discrete input $q \in \mathcal{Q}$. The rounding rule will chose the quantized input $q \in \mathcal{Q}$ if the relaxed input $u \in \mathcal{U}$ produced by the model predictive controller lies in its Voronoi cell $\mathcal{V}(q, \mathcal{Q})$.

Figure 1(c) shows the rounding errors produced by the rounding rule in Figure 1(a). Figure 1(c) shows the ellipsoidal set $\mathcal{E}(W^\star)$ produced by solving problem (10). Inside this ellipsoid are the shifted Voronoi cells $\mathcal{V}(q, \mathcal{Q}) - q$ for each discrete input $q \in \mathcal{Q}$. The rounding error set $\mathcal{W} = \mathrm{conv}\{\mathcal{V}(q, \mathcal{Q}) - q : q \in \mathcal{Q}\}$ is the convex-hull of the shifted Voronoi cells $\mathcal{V}(q, \mathcal{Q}) - q$. By Corollary 2, it is possible to solve the soft-landing problem with this rounding rule since the set of rounding errors $\mathcal{W}$ is contained $\mathcal{W} \subseteq \mathcal{E}(W^\star)$ in the ellipsoidal set $\mathcal{E}(W^\star)$. Thus it is possible to compute a robust control invariant set $\mathcal{C}$ for this problem. The robust control invariant set $\mathcal{C}$ is shown in Figure 2.



(a) Trajectories using rounding rule



(b) Trajectories using naive rounding rule

Fig. 2. Closed-loop position versus velocity of the transportation vehicle plotted on top of the projected state constraint set $\mathcal{X}$ and the projected maximal robust control invariant set $\mathcal{C}$.

The transportation vehicle was simulated in closed-loop with our model predictive controller and rounding rule. We used two difference tunings of the cost function (11a) in the model predictive controller; one that maximizes approach velocity and one that minimizes approach velocity. The simulation results are plotted in Figure 2 which shows the vehicle velocity versus position. In addition Figure 2 shows the projection of the state constraint set $\mathcal{X} \subseteq \mathbb{R}^3$ and maximal robust control invariant set $\mathcal{C} \subseteq \mathbb{R}^3$.

In each simulation trial, the model predictive controller uses the same nominal model of the dynamics, but the parameters of the actual dynamics of the transportation vehicle and input filter are randomly generated from the set possible parameters. Despite the model mismatch and input rounding, the controller was able to keep the state of the vehicle inside the state constraints as shown in Figure 2(a). Figure 2(b) shows the closed-loop trajectories when we use the naive rounding rule, shown in Figure 1(b), which chooses the quantized input $q \in \mathcal{Q}$ closest to the requested input $u \in \mathcal{U}$

$$\mathbf{q}(u) = \underset{q \in \mathcal{Q}}{\operatorname{argmin}} \|u - q\|_I^2 = \underset{q \in \mathcal{Q}}{\operatorname{argmin}} \; (u-q)^T(u-q).$$

In this rule the identity matrix is the weighting matrix $W = I$. For every trajectory simulated using this rounding rule, the closed-loop trajectory violated the constraints.

Figure 3 shows the input trajectories versus time for one simulation trial. The requested $u \in \mathcal{U}$ and implemented $q \in \mathcal{Q}$ inputs are often very different. The implemented input $q_a(u) \in \mathcal{Q}$ for the pneumatic actuator roughly tracks the requested input $u_a \in \mathcal{U}$ from the model predictive controller. However the implemented input $\mathbf{q}_e(u) \in \mathcal{Q}$ for the electric actuator is often very different from the requested value $u_e \in \mathcal{U}$. In fact, the implement input q_e for the electric actuator often has a different sign

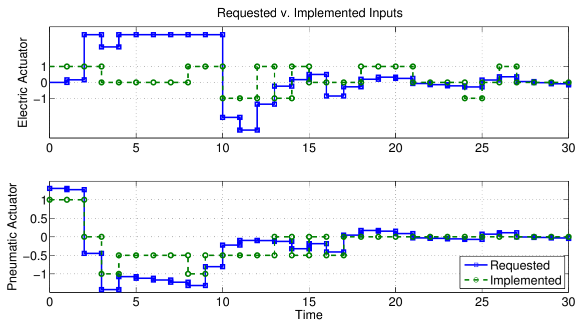


Fig. 3. An example of the requested input and implemented input for the pneumatic and electric actuators.

the requested input u_e depending on the requested input u_a to the pneumatic actuator. Despite the fact that the requested and implemented control inputs are very different in terms of Euclidean distance and sign, the resulting rounding errors are relatively easy for the system to reject without violating constraints.

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