MITSUBISHI ELECTRIC RESEARCH LABORATORIES
http://www.merl.com

Indirect Adaptive MPC for Output Tracking of Uncertain Linear Polytopic Systems

Zhou, J.; Di Cairano, S.; Danielson, C.

TR2017-056 May 2017

Abstract

We present an indirect adaptive model predictive control algorithm for output tracking of linear systems with polytopic uncertainty. The proposed approach is based on the velocity form of the system model, and achieves input-to-state stable output tracking with respect to the parameter estimation error and the rate of change of time-varying references. For the constrained case, recursive feasibility is achieved by including robust constraints designed from a robust control invariant set for the system model, and terminal constraints designed from a positive invariant set for the velocity model. Simulation results for a numerical example and an air conditioning control application demonstrate the method.

American Control Conference (ACC)

Indirect Adaptive MPC for Output Tracking of Uncertain Linear Polytopic Systems

Junqiang Zhou, Stefano Di Cairano†, Claus Danielson

Abstract— **We present an indirect adaptive model predictive control algorithm for output tracking of linear systems with polytopic uncertainty. The proposed approach is based on the velocity form of the system model, and achieves input-to-state stable output tracking with respect to the parameter estimation error and the rate of change of time-varying references. For the constrained case, recursive feasibility is achieved by including robust constraints designed from a robust control invariant set for the system model, and terminal constraints designed from a positive invariant set for the velocity model. Simulation results for a numerical example and an air conditioning control application demonstrate the method.**

I. INTRODUCTION

For controlling constrained systems subject to uncertainty, robust and adaptive model predictive control (MPC) methods have been proposed [1]. For uncertain systems modeled as polytopic linear difference inclusions (pLDIs), an indirect-adaptive MPC strategy was recently proposed [2], [3] that ensures input-to-state stability with respect to the parameter estimation error, robust constraint satisfaction, and computational burden similar to nominal MPC. The method was named indirect-adaptive MPC (IAMPC) because it adjusts the model and the cost function based on the parameter estimate obtained from an external estimator, which needs to satisfy only minimal assumptions. The estimate needs only to be a convex combination vector, resulting in the current prediction model being any convex combination of the vertices of the pLDI. Thus, IAMPC allows for separating controller and estimator design, which is desirable in practical applications.

The IAMPC in [2], [3] was developed for stabilization to a (fixed) equilibrium. However, several applications, require the controller to track time-varying (or at least piecewise constant) output references. Output tracking MPC [4], [5] is often based on first determining a target state and input setpoint (x_s, u_s), and then solving a state-input setpoint tracking problem, where the cost function penalizes the deviation from the setpoint,

$$V_N = F(x_{N|t}-x^s_{N|t})+\sum_{k=0}^{N-1} L(x_{k|t}-x^s_{k|t}, u_{k|t}-u^s_{k|t}). \quad (1)$$

In IAMPC, whenever the parameter estimate changes, the setpoint needs to be updated based on the adjusted model, and such changes are in general unpredictable. However, when the setpoint changes unpredictably from step to step, even if only for the last step of the prediction horizon, $(x^s_{N|t+1}, u^s_{N|t+1}) \neq (x^s_{N|t}, u^s_{N|t})$, the optimal cost (1) undergoes step changes. For such cases, the standard design methods, usually based on LMIs enforcing $F(\mathrm{d}x_{N|t+1}) + L(\mathrm{d}x_{N-1|t+1}, \mathrm{d}u_{N-1|t+1}) - F(\mathrm{d}x_{N|t}) \leq 0$ where $\mathrm{d}a = a - a^s$, may not guarantee the decrease of the optimal cost even in the perfect model case, because the setpoints in the different terms are actually not equal. A method to overcome some of these limitations is to formulate the prediction model in velocity form [6]–[8], which avoids the need of explicitly computing the state and input setpoint by controlling instead the state variation and the output tracking error.

In this paper we exploit the velocity model together with a set of integrators that reconstruct the system state and input. We propose a design for the terminal cost based on the velocity model that achieves input-to-state stable output tracking with respect to both the parameter estimation error and the reference change. The relation between the velocity model and the system model is exploited to design invariant sets that guarantee recursive feasibility. The system model is used to determine a robust control invariant set which ensures constraint satisfaction even in presence of uncertainty. The terminal set is designed based on a positive invariant set obtained by considering together the velocity model and the system model. The resulting IAMPC for output tracking solves a quadratic program that is only slightly larger than the one for stabilization, where the increase is only due to the higher dimensional (usually more complex) terminal set.

In what follows, the system model with state and input constraints and the equivalent velocity form are introduced in Section II, and the IAMPC strategy and the problem definition are described in Section III. The design methods for the unconstrained and constrained IAMPC are proposed in Sections IV and V, respectively. Simulations on an air conditioning case study are reported in Section VI, and conclusions in Section VII. Due to the limited space, only brief descriptions of the proofs are provided.

Notation: $\mathbb{R}, \mathbb{R}_{0+}, \mathbb{R}_+, \mathbb{Z}, \mathbb{Z}_{0+}, \mathbb{Z}_+$ denote the set of real, nonnegative real, positive real, and integer, nonnegative integer, positive integer numbers. We denote intervals by notations such as $\mathbb{Z}_{[a,b)} = \{z \in \mathbb{Z} : a \leq z < b\}$. For vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $(x, y) = [x' \ y']'$, and $[x]_i$ is the i-th component. For $x \in \mathbb{R}^n$, $|x|_p$ is the p-norm, and $|x|$ the Euclidean norm, whereas for $A \in \mathbb{R}^{n\times n}$, $|A|_p$ is the induced p-norm, and $|A| = |A|_2$. For $\phi : \mathbb{Z}_{0+} \to \mathbb{R}^n$, $\|\phi\| = \sup\{|\phi(t)| : t \in \mathbb{Z}_{0+}\}$. A continuous-time signal $x(\tau)$

J. Zhou was an intern at Mitsubishi Electric Research Laboratories during this research. He is now with GE Global Research, Niskayuna, NY, junqiang.zhou@ge.com.

S. Di Cairano and C. Danielson are with Mitsubishi Electric Research Laboratories, Cambridge, MA, dicairano@ieee.org, danielson@merl.com . † Corresponding author.

sampled with a period T_s is denoted by the discrete-time signal $x(t) = x(tT_s)$, where $t \in \mathbb{Z}_{0+}$. For $x(t) \in \mathbb{R}^n$, $x_{k|t}$ denotes the value predicted k-step predicted from $x(t)$, and $x_{[0,N|t]} = (x_{0|t}, \dots, x_{N|t})$. For $\Xi \subseteq \mathbb{R}^\ell$, $\Xi^M = \Xi \times \cdots \times \Xi$.

II. System Model and Velocity Form

We consider the constrained linear discrete-time system with polytopic uncertainty described by

$$x(t+1) = \sum_{i=1}^{\ell} [\bar{\xi}]_i A_i x(t) + Bu(t), \quad x(0) = x_0 \tag{2a}$$

$$y(t) = Cx(t) \tag{2b}$$

$$x(t) \in \mathcal{X}, \quad u(t) \in \mathcal{U}, \tag{2c}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $y \in \mathbb{R}^p$. The constraint sets on states $\mathcal{X} \subseteq \mathbb{R}^n$ and inputs $\mathcal{U} \subseteq \mathbb{R}^m$ are polyhedral. The uncertain state matrix $A(\bar{\xi}) := \sum_{i=1}^{\ell} [\bar{\xi}]_i A_i$ is a convex combination of given vertex matrices $A_i \in \mathbb{R}^{n\times n}, i \in \mathbb{Z}_{[1,\ell]}$, with unknown convex combination vector $\bar{\xi} \in \mathbb{R}^\ell$, $\bar{\xi} \in \Xi = \{\xi \in \mathbb{R}^\ell : 0 \le [\xi]_i \le 1, \sum_{i=1}^{\ell} [\xi]_i = 1\}$.

We want (2) to track a time-varying reference $r \in \mathbb{R}^p$,

$$r(t+1) = r(t) + \Delta r(t), \quad r(0) = r_0, \tag{3}$$

where $\Delta r(t) \in \mathbb{R}^p$ is such that the reference signal evolves within an admissible set, $r(t) \in \Omega_r \subseteq \mathbb{R}^p$. Piecewise constant references are included in (3) by Δr being *almost* always 0. In some cases the admissible reference set Ω_r may not be pre-assigned, and must be determined with the controller. Since $r(t) \in \Omega_r$, it holds that $\Delta r(t) \in \Omega_{\Delta r}(r(t))$, where $\Omega_{\Delta r}(r) = \{\Delta r \in \mathbb{R}^p : r + \Delta r \in \Omega_r\}$.

Assumption 1:
1) The state x is measured;
2) The plant is input-output square, i.e., $m = p$;
3) The linear polytopic system (2) is reversible, that is, state matrix $A(\bar{\xi})$ is non-singular for any $\bar{\xi} \in \Xi$;
4) $(A(\bar{\xi}), B)$ is controllable, and $(C, A(\bar{\xi}))$ is observable for any $\bar{\xi} \in \Xi$.
5) For any $\bar{\xi} \in \Xi$, the linear polytopic system (2) has no invariant zeros on the unit circle, that is,

$$\operatorname{rank} \begin{bmatrix} A(\bar{\xi}) - I_n & B \\ C & 0 \end{bmatrix} = n + p, \quad \forall \bar{\xi} \in \Xi. \tag{4}$$

□

Assumption 1.5 implies that for any uncertain parameter $\bar{\xi} \in \Xi$, given a reference $r_s \in \mathbb{R}^p$, there exists a unique steady state pair $(x_s, u_s) \in \mathbb{R}^{n+p}$ that satisfies

$$x_s = A(\bar{\xi})x_s + Bu_s, \qquad r_s = Cx_s, \tag{5}$$

where $x_s = x_s(\bar{\xi}), u_s = u_s(\bar{\xi})$ are functions of $\bar{\xi}$.

In what follows, the value of $\bar{\xi}$ in (2) is not known. As opposed to considering the uncertainty as an additive disturbance and thus applying robust MPC, such as in [7] where the velocity form was used, here an estimator provides a time-varying estimate $\xi(t)$ of $\bar{\xi}$, such that $\xi(t) \in \Xi$ for all $t \in \mathbb{Z}_{0+}$, which is then used in an adaptive MPC. The need to update the equilibrium state and input targets is avoided by using the velocity form of (2).

A. Velocity Model Formulation

To construct the velocity form of (2), we introduce the input increment $\Delta u(t) \in \mathbb{R}^p$ and formulate the input as

$$v(t+1) = v(t) + \Delta u(t), \quad v(0) = v_0, \tag{6}$$

where $v(t) = u(t-1)$. Then, as in [7], [8], we define the state increment $\Delta x(t) = x(t) - x(t-1)$. The output tracking error is $e(t+1) = y(t+1) - r(t+1) = e(t) + C\Delta x(t+1) - \Delta r(t)$. For a constant $\bar{\xi} \in \Xi$, the velocity form of (2a)-(2b) with state $\delta(t) = (\Delta x(t), e(t))$, denoted as $\Sigma_1(\bar{\xi})$, is

$$\delta(t+1) = A_v(\bar{\xi})\delta(t) + B_v \Delta u(t) + G_v \Delta r(t), \delta(0) = \delta_0 \tag{7a}$$

$$e(t) = C_v \delta(t). \tag{7b}$$

The following proposition follows immediately from the system structure.

Proposition 1: Under Assumption 1.3 and 1.4, the pair $(A_v(\bar{\xi}), B_v)$ is controllable and the pair $(C_v, A_v(\bar{\xi}))$ is observable for any $\bar{\xi} \in \Xi$. □

While [7] exploits the exact knowledge of the model to derive a static relationship by inversion, to enforce the constraints in (2c), we formulate another system, $\Sigma_2(\bar{\xi})$,

$$\phi(t+1) = \phi(t) + E_v(\bar{\xi})\delta(t) + F_v \Delta u(t), \phi(0) = \phi_0 \tag{7c}$$

$$z(t) = \phi(t) + H_v \Delta u(t), \tag{7d}$$

where the state is $\phi(t) = (x(t), v(t))$. $\Sigma_2(\bar{\xi})$ is a discrete-time integrator for $\Sigma_1(\bar{\xi})$. The system matrices in (7) are

$$A_{v_i} = \begin{bmatrix} A_i & 0 \\ CA_i & I_p \end{bmatrix}, B_v = \begin{bmatrix} B \\ CB \end{bmatrix}, G_v = \begin{bmatrix} 0 \\ -I_p \end{bmatrix}, \tag{8}$$

$$C_v = \begin{bmatrix} 0 & I_p \end{bmatrix}, E_{v_i} = \begin{bmatrix} A_i & 0 \\ 0 & 0 \end{bmatrix}, F_v = \begin{bmatrix} B \\ I_p \end{bmatrix}, H_v = \begin{bmatrix} 0 \\ I_p \end{bmatrix}.$$

The collection of equations in (7) describe the integrator-augmented velocity model $\Sigma(\bar{\xi})$ to be used as IAMPC prediction model,

$$\chi(t+1) = A_a(\bar{\xi})\chi(t) + B_a \Delta u(t) + G_a \Delta r(t), \ \chi(0) = \chi_0, \tag{9a}$$

$$e(t) = C_a \chi(t), \qquad z(t) = D_a \chi(t) + E_a \Delta u(t), \tag{9b}$$

where $\chi(t) = (\delta(t), \phi(t))$ is the state, $e(t)$ is the performance output and $z(t) = (x(t), u(t))$ is the constrained output.

Remark 1: For the trajectories of (2) and integrator-augmented velocity model (9) to be equivalent, the initial condition Δx_0 in χ_0 must be properly determined. As discussed in [6] for a fixed $\bar{\xi} \in \Xi$, the initial condition Δx_0 needs to satisfy

$$(A(\bar{\xi}) - I)x_0 - A(\bar{\xi})\Delta x_0 + Bv_0 = 0, \tag{10}$$

For any sequence of input increments Δu in (6) and initial condition $(x_0, v_0) \in \mathcal{X} \times \mathcal{U}$, if (10) holds, the trajectories for $x(t)$ generated by (2) and (9) are equal. □

In MPC, the initial state of the prediction model is re-initialized at each sampling time t. By selecting $\chi(t) = (x(t) - x(t-1), Cx(t) - r(t), x(t), u(t-1))$, where $x(t-1), u(t-1)$ are the signals from the previous sampling instant, condition (10) is satisfied. Therefore, (9) can be used to predict trajectories of (2).

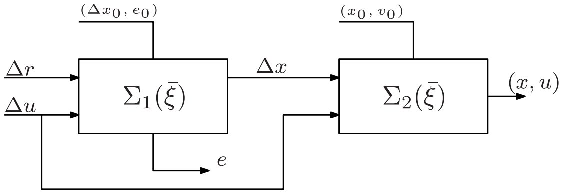

Fig. 1. Cascaded structure of the constrained velocity model: $\Sigma_1(\bar{\xi})$ includes the velocity states $(\Delta x, e)$ and $\Sigma_2(\bar{\xi})$ includes the constrained states (x, v).

B. Velocity Model Structure

The structure of integrator-augmented velocity model (9), in which the state Δx of $\Sigma_1(\bar{\xi})$ becomes an input of $\Sigma_2(\bar{\xi})$, is shown in Fig. 1. The Δu–e relationship reveals that the integrator-augmented velocity model (9) is in Kalman observable decomposition, where δ is observable and controllable, while ϕ of $\Sigma_2(\bar{\xi})$ is non-observable with respect to e. Because of the cascade coupling and since $\Sigma_2(\bar{\xi})$ models the constrained variables, $\Sigma_1(\bar{\xi})$ needs to be controlled such that the trajectory of $(\Delta x(t), \Delta u(t))$ ensures that $(x(t), u(t))$ in $\Sigma_2(\bar{\xi})$ satisfies the constraints. For achieving output tracking, only part of the state of $\Sigma_1(\bar{\xi})$, i.e., e, needs to vanish. By (5), tracking a constant reference implies convergence of (x, u) to a unique (x_s, u_s). Thus, when tracking a constant reference, at steady state Δu and Δx asymptotically vanish, and the controller actually asymptotically stabilizes the entire state $\delta(t) = (\Delta x(t), e(t))$ of $\Sigma_1(\bar{\xi})$.

III. Indirect Adaptive Model Predictive Control

The output tracking IAMPC exploits prediction model (9). However, since $\bar{\xi}$ is not known, a (time-varying) estimate $\xi(t)$ is used in the prediction model. The parameter estimation error is $\tilde{\xi}(t) = \bar{\xi} - \xi(t)$, and $\tilde{\xi}(t) \in \tilde{\Xi}(\xi(t))$ where $\tilde{\xi}(t) \in \tilde{\Xi}(\xi) = \{\tilde{\xi} \in \mathbb{R}^\ell : \exists \bar{\xi} \in \Xi, \text{ s.t. } \tilde{\xi} = \bar{\xi} - \xi\}$. The following assumption on the parameter estimate $\xi(t)$, and on the uncertain parameter values used by IAMPC for prediction $\xi_{k|t}$, is made throughout.

Assumption 2: The estimate provided to the controller is such that $\xi(t) \in \Xi$ for all $t \in \mathbb{Z}_{0+}$. At any time $t \in \mathbb{Z}_+$, the predicted value of the parameter estimate $\xi_{k|t}, k \in \mathbb{Z}_{[0,N]}$, is such that $\xi_{k|t} = \xi_{k+1|t-1}$ for all $k \in \mathbb{Z}_{[0,N-1]}$. □

At time t, given $\chi(t)$ and the predicted parameter vector $\xi_{[0,N|t]}$, the IAMPC based on (9) solves

$$V_N^\star(\chi(t), \xi_{[0,N|t]}) = \min_{\Delta \mathbf{u}(t)} V_N(\Delta \mathbf{u}(t);\ \chi(t), \xi_{[0,N|t]}) \quad \text{(11a)}$$

$$= \min_{\Delta \mathbf{u}(t)} \delta'_{N|t} P(\xi_{N|t}) \delta_{N|t} + \sum_{k=0}^{N-1} \delta'_{k|t} Q \delta_{k|t} + \Delta u'_{k|t} R \Delta u_{k|t}$$

$$\text{s.t.} \quad \delta_{k+1|t} = A_v(\xi_{k|t})\delta_{k|t} + B_v \Delta u_{k|t} \quad \text{(11b)}$$

$$\phi_{k+1|t} = \phi_{k|t} + E_v(\xi_{k|t})\delta_{k|t} + F_v \Delta u_{k|t}, \quad \text{(11c)}$$

$$z_{k|t} = \phi_{k|t} + H_v \Delta u_{k|t} \in \mathcal{C}_{xu}, \quad \text{(11d)}$$

$$(\delta_{N|t}, \phi_{N|t}) \in \mathcal{X}_\chi^f, \quad \text{(11e)}$$

$$\delta_{0|t} = \delta(t),\ \phi_{0|t} = \phi(t) \quad \text{(11f)}$$

where $N \in \mathbb{Z}_+$ is the prediction horizon, $Q \in \mathbb{R}^{(n+p)\times(n+p)}$ and $R \in \mathbb{R}^{m\times m}$, $Q, R \succ 0$. The prediction is based on a "nominal model" (11b)-(11d) of the integrator-augmented velocity model (9) with time-varying prediction parameter $\xi_{k|t}$ and constant reference along the horizon, $\Delta r_{k|t} = 0$. The design of the terminal cost $P(\xi) \in \mathbb{R}^{(n+p)\times(n+p)}$, constraint set $\mathcal{C}_{xu} \subseteq \mathcal{X} \times \mathcal{U}$ and terminal set $\mathcal{X}_\chi^f \subset \mathbb{R}^{2(n+p)}$ is derived in the subsequent sections. $\Delta \mathbf{u}(t) = (\Delta u_{0|t}, \ldots, \Delta u_{N-1|t})$ is the optimizer and $\Delta \mathbf{u}^\star(t) = (\Delta u^\star_{0|t}, \ldots, \Delta u^\star_{N-1|t})$ is the optimal solution to (11). The control input at sampling time t is determined from the first input increment $\Delta u(t) = \Delta u^\star_{0|t} = K^{\text{MPC}}(\chi(t), \xi_{[0,N|t]})$, resulting in the IAMPC law

$$u(t) = u(t-1) + K^{\text{MPC}}(\chi(t), \xi_{[0,N|t]}) \quad (12)$$

Solving (11) only requires the solution of a quadratic program, similar to a nominal MPC.

As discussed in [2], Assumption 2 can be satisfied by obtaining $\xi(t)$ as the projection of any estimate $\varrho(t)$ onto Ξ, and by adjusting the parameter value only at the end of the prediction horizon, $\xi_{t|N}$ while shifting the previous values for the other steps so that $\xi_{k|t} = \xi_{k+1|t-1}$ for all $k \in \mathbb{Z}_{[0,N-1]}$. This amounts to including a N-step delay in the estimate update, $\xi_{k|t} = \xi(t-N+k)$. Next, we state a useful property of the value function $V_N^\star(\chi(t), \xi_{[0,N|t]})$.

Lemma 1: Let $\mathcal{C}_{xu}$ and $\mathcal{X}_\chi^f$ be polyhedral, $\chi(t) \in \mathcal{S}_\chi$, and $\mathcal{S}_\chi$ be a bounded set in which the optimization problem (11) is feasible. Then, the value function $V_N^\star(\chi, \xi_{[0,N|t]})$ is Lipschitz continuous in $\mathcal{S}_\chi$, that is, there exists $L \in \mathbb{R}_+$ such that for all $\xi_{[0,N|t]} \in \Xi^{N+1}$, and for all $\chi_1, \chi_2 \in \mathcal{S}_\chi$

$$|V_N^\star(\chi_1, \xi_{[0,N|t]}) - V_N^\star(\chi_2, \xi_{[0,N|t]})| \leq L|\chi_1 - \chi_2|. \quad (13)$$

□

The proof for Lemma 1 is based on $V_N^\star(\chi, \xi_{[0,N|t]})$ being piecewise quadratic [9]. Additional steps can be found in [3]

Next, we define the problem to be solved by the control design.

Problem 1: Consider system (2) subject to Assumption 1 and the associated integrator-augmented velocity model (9), a time-varying reference (3), a parameter estimator producing $\xi(t)$ and $\xi_{[0,N|t]}$ satisfying Assumption 2 for any $t \in \mathbb{Z}_{0+}$, design the IAMPC law (12) based on (11) such that:

1) the closed-loop $\Sigma_1(\bar{\xi})$ is input-to-state-stable (ISS) with respect to the estimation error and the reference change, i.e., there exits a $\mathcal{KL}$ function β and $\mathcal{K}$ functions ρ_1, ρ_2 such that for any $t \in \mathbb{Z}_{0+}$

$$|\delta(t)| \leq \beta(|\delta_0|, t) + \rho_1(\|\tilde{\xi}_{0|t}\|) + \rho_2(\|\Delta r\|), \quad (14)$$

2) the constraints on $\Sigma_2(\bar{\xi})$ are satisfied, i.e., there exists a set $\mathcal{X}_{dxu} \subseteq \mathbb{R}^n \times \mathcal{X} \times \mathcal{U}$ such that for every $(\Delta x(0), x(0), v(0)) \in \mathcal{X}_{dxu}$ and any $r(t) \in \Omega_r$, for all $t \in \mathbb{Z}_{0+}$, $(x(t), u(t)) \in \mathcal{X} \times \mathcal{U}$, for every $t \in \mathbb{Z}_{0+}$. □

In Problem 1, (14) ensures bounded tracking error for time-varying references. Furthermore,, when $\lim_{t\to\infty} \tilde{\xi}(t) = 0$ and $\lim_{t\to\infty} r(t) = r_s$, $\Sigma_1(\bar{\xi})$ is asymptotically stable. For estimation errors that are non-vanishing, yet are eventually smaller than a certain threshold, $|\tilde{\xi}(t)| \leq \bar{\eta}$, for all $t > \bar{t}$, asymptotic stability can be obtained following [3].

IV. Unconstrained Tracking IAMPC

First, we consider the unconstrained case, where, with slight abuse of notation and reminding that $m = p$, $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^p$, and the corresponding invariant sets are $\mathcal{C}_{xu} = \mathbb{R}^{n+p}$ and $\mathcal{X}_\chi^f = \mathbb{R}^{2(n+p)}$.

Based on (9), we first design a stabilizing control for the nominal linear parameter varying system $\delta(t+1) = \sum_{i=1}^{\ell}[\xi(t)]_i A_{v_i}\delta(t) + B_v\Delta u(t)$ where $\xi(t) \in \Xi$.

Consider the parameter-dependent Lyapunov function [10]

$$V_f(\delta, \xi) = \delta' P(\xi)\delta, \quad P(\xi) = \sum_{i=1}^{\ell}[\xi]_i P_i, \tag{15}$$

where $P_i \succ 0$ for all $i \in \mathbb{Z}_{[1,\ell]}$, with the associate stabilizing control law $\Delta u = K(\xi)\delta$, where $K(\xi) = \sum_{i=1}^{\ell}[\xi]_i K_i$, such that the closed-loop system satisfies

$$\begin{aligned} V_f\left(\left(A_v(\xi) + B_v K(\xi)\right)\delta, \xi^+\right) - V_f(\delta, \xi) \\ \leq -\delta'\left(Q + K(\xi)' R K(\xi)\right)\delta, \ \forall\, \xi, \xi^+ \in \Xi. \end{aligned} \tag{16}$$

LMIs for constructing $K(\xi)$ and $P(\xi)$ that satisfy (16). can be found for instance in [2], [10].

Next, we analyze ISS with respect $\tilde{\xi}(t) \in \tilde{\Xi}(\xi(t))$ and $\Delta r(t) \in \Omega_{\Delta r}(r(t))$ of (9) in closed-loop with the unconstrained IAMPC where the terminal cost in (11a) satisfies (16). Since $\Sigma_2(\bar{\xi})$ does not affect the cost function (11a), in the unconstrained case only $\Sigma_1(\bar{\xi})$ is considered [8]. The next result follows from arguments in [9, Ch.2] and [2] for unconstrained MPC for linear time-varying systems.

Proposition 2: Consider the IAMPC (12) when $\mathcal{X} = \mathbb{R}^n, \mathcal{U} = \mathbb{R}^p$, $\mathcal{C}_{xu} = \mathbb{R}^n \times \mathbb{R}^p$, $\mathcal{X}_\chi^f = \mathbb{R}^{2(n+p)}$. The value function of (11), $V_N^\star(\chi(t), \xi_{[0,N|t]})$ is such that there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}_+$ for which

$$\alpha_1|\delta(t)|^2 \leq V_N^\star(\chi(t), \xi_{[0,N|t]}) \leq \alpha_2|\delta(t)|^2, \tag{17a}$$

$$V_N^\star(\chi_{1|t}, \xi_{[0,N|t+1]}) - V_N^\star(\chi(t), \xi_{[0,N|t]}) \leq -\alpha_3|\delta(t)|^2, \tag{17b}$$

where $\chi_{1|t} = A_a(\xi(t))\chi(t) + B_a K^{\text{MPC}}(\chi(t), \xi_{[0,N|t]})$. □

Proposition 2 relates $\chi(t)$ and $\chi_{1|t}$, and hence holds for a "nominal" system, and follows from arguments in [9, Ch.2] and [2] for MPC for linear time-varying systems.

Theorem 1: For system (2) and the associated integrator-augmented velocity model (9), let $\mathcal{X} = \mathbb{R}^n$, $\mathcal{U} = \mathbb{R}^p$. Consider the IAMPC law (12) that solves (11), where $P(\xi)$ satisfies (16), $\mathcal{C}_{xu} = \mathbb{R}^{n+p}$ and $\mathcal{X}_\chi^f = \mathbb{R}^{2(n+p)}$. In any bounded set, the value function $V_N^\star(\chi(t), \xi_{[0,N|t]})$ is an ISS-Lyapunov function for $\Sigma_1(\bar{\xi})$ with respect to the estimation error $\tilde{\xi}_{0|t} \in \tilde{\Xi}(\xi_{0|t})$ and reference change $\Delta r(t) \in \Omega_{\Delta r}(r(t))$ for the close loop, and there exists $\alpha, \gamma, \rho \in \mathbb{R}_+$ such that

$$\begin{aligned} V_N^\star(\chi(t+1), \xi_{[0,N|t+1]}) - V_N^\star(\chi(t), \xi_{[0,N|t]}) \\ \leq -\alpha|\delta(t)|^2 + \gamma|\tilde{\xi}_{0|t}|^2 + \rho|\Delta r(t)|. \end{aligned} \tag{18}$$

□

The proof of Theorem 1 is based on Proposition 2, that guarantees a decrease of the value function when there is error and the reference does not change, and Lemma 1 that allows to bound the effect of the error and reference change.

Theorem 1 guarantees output tracking since the tracking error is a state variable in $\Sigma_1(\bar{\xi})$, and (9) and (2) generate equal trajectories under (10), which holds for IAMPC. Due to the structure of (9), when $|\tilde{\xi}_{0|t}| = |\Delta r(t)| = 0$, for all $t > \bar{t}$, by Assumption 1 the convergence of the output to a constant reference implies the convergence of the state and input of (2) to a steady state. As for Lyapunov stability, this is only established for the tracking error and the step-to-step change of state and input of (2), which is in general sufficient in output tracking applications.

V. Constrained Tracking IAMPC

Next, we consider the case when system (2) is subject to constraints, i.e., $\mathcal{X} \subset \mathbb{R}^n, \mathcal{U} \subset \mathbb{R}^p$ and $\mathcal{X}, \mathcal{U}$ are compact sets. In this case, for ensuring constraint satisfaction and retaining stability, we need to design the robust constraints (11d) and the terminal set (11e).

A. Robust Constraints and Terminal Set Design

1) Robust Constraint Set: To guarantee robust constraint satisfaction in the presence of parameter estimation error, the constraint set $\mathcal{C}_{xu}$ is designed based on a robust control invariant (RCI) set [11] of (2). Let $\mathcal{C}_x \subseteq \mathcal{X}$ be the RCI set such that for any $x \in \mathcal{C}_x$ there exists $u \in \mathcal{U}$ that satisfies $A_i x + Bu \in \mathcal{C}_x$ for all $i \in \mathbb{Z}_{[1,\ell]}$. Then, $\mathcal{C}_{xu}$ in (11) is

$$\mathcal{C}_{xu} = \{(x, u) \in \mathcal{C}_x \times \mathcal{U} \ : \ A_i x + Bu \in \mathcal{C}_x, \forall i \in \mathbb{Z}_{[1,\ell]}\}. \tag{19}$$

If (11d) is designed as in (19), whenever (11) is feasible, the constraints (2c) are satisfied despite the estimation error.

2) Terminal Constraint Set: The terminal set $\mathcal{X}_\chi^f$ is designed as an invariant constraint admissible set for (9), which couples the state δ of $\Sigma_1(\bar{\xi})$ and state ϕ of $\Sigma_2(\bar{\xi})$. Let $K(\xi)$ be the control law that satisfies (16). Let $K_a(\xi) = \begin{bmatrix} K(\xi) & 0_{p\times(n+p)} \end{bmatrix}$ be its extension to $\Sigma(\bar{\xi})$, and $\mathcal{X}_K = \{\chi : (D_a + E_a K_{a_i})\chi \in \mathcal{C}_{xu}, \ i \in \mathbb{Z}_{[1,\ell]}\}$. Then, $\mathcal{X}_\chi^f \subseteq \mathcal{X}_K$ is robust positive invariant (RPI) with respect to the time-varying parameter ξ, and contained in $\mathcal{X}_K$,

$$\mathcal{X}_\chi^f = \{\chi \subseteq \mathcal{X}_K : \ (A_{a_i} + B_a K_{a_i})\chi \in \mathcal{X}_\chi^f, \ \forall i \in \mathbb{Z}_{[1,\ell]}\} \tag{20}$$

$\mathcal{X}_\chi^f$ is RPI with respect to known changes of ξ, but in general is not RPI in the presence of a parameter estimation error. Thus, recursive feasibility of the terminal constraint in presence of such error is achieved by properly choosing the length of the prediction horizon as explained next.

B. Prediction Horizon and Domain of Attraction

Satisfaction of (11e) for $\mathcal{X}_\chi^f \neq \mathbb{R}^{2(n+p)}$ in the presence of estimation error requires $\mathcal{X}_\chi^f$ to be reachable from every allowed state in N steps. Thus, one can achieve recursive feasibility by first determining the set of allowed initial states of (11), and then determining N so that the terminal set is N-step reachable from any of those.

Next, first we determine the actual set of allowed states. Then, since such set depends on the unknown parameter $\bar{\xi}$, we construct a parameter-independent outer approximation.

Finally, the horizon is selected to ensure reachability of the terminal set from any state in the outer approximation.

To simplify the sets construction, we apply a change of coordinates to $\Sigma_1(\bar{\xi})$ defining as the new state $\hat{\delta}(t) = (\Delta x(t), r(t))$. This results in

$$\hat{\chi}(t+1) = \hat{A}_a(\bar{\xi})\hat{\chi}(t) + \hat{B}_a\Delta u(t) + \hat{G}_a\Delta r(t), \hat{\chi}(0) = \hat{\chi}_0,$$
$$e(t) = \hat{C}_a\hat{\chi}(t), \qquad z(t) = \hat{D}_a\hat{\chi}(t) + \hat{E}_a\Delta u(t), \quad (21)$$

where $\hat{\chi} = (\hat{\delta}, \phi) = g(\chi)$ and g is, with a small abuse of notation, the change of coordinates.

The set of admissible states is determined as follows.

Lemma 2: For any $\bar{\xi} \in \Xi$,

$$\hat{\mathcal{S}}_\chi(\bar{\xi}) = \{(\Delta x, r, x, v) : r \in \Omega_r, x \in \mathcal{C}_x, v \in \mathcal{U}, (A(\bar{\xi}) - I)x - A(\bar{\xi})\Delta x + Bv = 0\}. \quad (22)$$

is a control invariant set for (21), i.e., for all $\hat{\chi} \in \hat{\mathcal{S}}_\chi(\bar{\xi})$, there exists Δu such that $\hat{A}_a(\bar{\xi})\hat{\chi} + \hat{B}_a\Delta u + \hat{G}_a\Delta r \in \hat{\mathcal{S}}_\chi(\bar{\xi})$. □
The proof of Lemma 2 is based on invariance of $\mathcal{C}_x$ and the equivalence of trajectories of (2) and (9) under the initialization condition (10).

By (22), the components (r, x, v) of $(\Delta x, r, x, v)$ in $\hat{\mathcal{S}}_\chi(\bar{\xi})$ do not depend directly on $\bar{\xi}$. Thus, the allowed values for (r, x, v) are determined only by the corresponding constraints. On the other hand, under Assumption 1, given (r, x, v), there is a unique value of Δx that satisfies (22), and it depends on $\bar{\xi}$ in an usually nonlinear way. Because of this $\hat{\mathcal{S}}_\chi(\bar{\xi})$ cannot in general be explicitly computed and a parameter-independent outer approximation is necessary.

Hence, we build $\hat{\mathcal{S}}_\chi \supseteq \hat{\mathcal{S}}_\chi(\bar{\xi})$ for all $\bar{\xi} \in \Xi$,

$$\hat{\mathcal{S}}_\chi = \{(\Delta x, r, x, v) \in \mathbb{R}^n \times \Omega_r \times \mathcal{C}_x \times \mathcal{U} : \exists \Delta u, \quad (23)$$
$$(x, v + \Delta u) \in \mathcal{C}_{xu},\, x + A_i\Delta x + B\Delta u \in \mathcal{C}_x, \forall i \in \mathbb{Z}_{[1,\ell]}\}.$$

Lemma 3: Let $\hat{\mathcal{S}}_\chi(\bar{\xi})$ and $\hat{\mathcal{S}}_\chi$ be defined as in (22) and (23), respectively. Then, $\hat{\mathcal{S}}_\chi(\bar{\xi}) \subseteq \hat{\mathcal{S}}_\chi$ for any $\bar{\xi} \in \Xi$. □
The proof of Lemma 3 is based on showing that for any $\bar{\xi}$, $\hat{\chi} \in \hat{\mathcal{S}}_\chi(\bar{\xi})$ implies that $\hat{\chi} \in \hat{\mathcal{S}}_\chi$. This is due to $\hat{\mathcal{S}}_\chi(\bar{\xi})$ being defined by an invariance condition and the initialization (10), while $\hat{\mathcal{S}}_\chi$ is defined by invariance of the system and its velocity form. Any state that satisfies the invariance of (2) and (10) also satisfies the invariance of the velocity form, due to equivalence. Yet, some states may satisfy both invariance conditions, without satisfying the initialization.

By $\hat{\mathcal{S}}_\chi$, we obtain a design for the prediction horizon.

Lemma 4: Let $h^* \in \mathbb{Z}_+$ be the smallest value such that $\hat{\mathcal{R}}_\chi^{h^*} \supseteq \hat{\mathcal{S}}_\chi$, where $\hat{\mathcal{R}}_\chi^{(h)}$ is the set of states that for any given $\xi_{[0,h-1]} \in \Xi^h$ can be brought to $\hat{\mathcal{R}}_\chi^{(0)} = \hat{\mathcal{X}}_\chi^f$ in h steps while satisfying the constraints (11d), and is computed as

$$\hat{\mathcal{R}}_\chi^{(0)} = \hat{\mathcal{X}}_\chi^f$$
$$\hat{\mathcal{R}}_\chi^{(h+1)} = \bigcap_{i \in \mathbb{Z}[1,\ell]} \Big\{\hat{\chi} : \exists \Delta u,\ \hat{A}_{a_i}\hat{\chi} + \hat{B}_a\Delta u \in \hat{\mathcal{R}}_\chi^{(h)}, \hat{D}_a\hat{\chi} + \hat{E}_a\Delta u \in \mathcal{C}_{xu}\Big\}. \quad (24)$$

Then, the IAMPC law (12) where (11d) is designed according to (19), $P(\xi)$ in (11a) satisfies (16), (11e) is designed according to (20), and $N \geq h^*$, is recursively feasible within $\mathcal{X}_\chi^{\mathcal{F}} = \{\chi :\ g(\chi) = \hat{\chi} \in \hat{\mathcal{S}}_\chi\}$, even in the presence of estimation error $\tilde{\xi}_{0|t} \in \Xi(\xi_{0|t})$. □

The proof of Lemma 4 is based on showing that (11d) designed by (19) ensures satisfaction of (2c) and that the terminal constraint is feasible because for the proposed choice of N, the terminal set $\mathcal{X}_\chi^{\mathcal{F}}$ is reachable within the horizon from every χ such that $g(\chi) = \hat{\chi} \in \hat{\mathcal{S}}_\chi$.

In computing $\hat{\mathcal{R}}_\chi^{(h)}$ at each step we allow for a different command for each vertex system, i.e., the controller is aware of ξ, as opposed to (19), where the same input is applied to all vertices. Thus results in $\hat{\mathcal{R}}_\chi^{(h)}$ being larger than the corresponding robust reachable set and ensures that we can find h^* such that $\hat{\mathcal{R}}_\chi^{(h)} \supseteq \mathcal{S}_\chi(\bar{\xi})$. Furthermore, $\hat{\mathcal{S}}_\chi = \{(x, \Delta x, v) :\ \exists r \in \Omega_r, (x, r, \Delta x, v) \in \hat{\mathcal{S}}_\chi\} \times \Omega_r$. This allows for searching for a suitable Ω_r during the iteration (24), for trading off between Ω_r and the prediction horizon N, and possibly for combining IAMPC with the techniques in [5].

C. Input-to-State Stability

Next we combine the results of Section IV with the design of the constraint sets and the prediction horizon.

Theorem 2: Consider system (2), and the IAMPC law (12) that solves (11), with prediction model (9), $P(\xi)$ that saisfies (16), $\mathcal{C}_{xu}$ and $\mathcal{X}_\chi^f$ designed according to (19) and (20), respectively, and N selected according to Lemma 4. The value function $V_N^\star(\chi(t), \xi_{[0,N|t]})$ of (11) is an ISS-Lyapunov function for $\delta = (\Delta x, e)$ within the invariant set $\mathcal{X}_\chi^{\mathcal{F}}$, with respect to the estimation error $\tilde{\xi}_{0|t} \in \tilde{\Xi}(\tilde{\xi}_{0|t})$ and reference rate $\Delta r(t) \in \Omega_{\Delta r}(r(t))$. Thus, there exists $\alpha, \gamma, \rho \in \mathbb{R}_+$ such that for all $\chi(t) \in \mathcal{X}_\chi^{\mathcal{F}}$,

$$V_N^\star(\chi(t+1), \xi_{[0,N|t+1]}) - V_N^\star(\chi(t), \xi_{[0,N|t]}) \leq -\alpha|\delta(t)|^2 + \gamma|\tilde{\xi}_{0|t}|^2 + \rho|\Delta r(t)|. \quad (25)$$

□

The proof of Theorem 2 follows by combining Theorem 1 that guarantees ISS in the unconstrained case, with Lemma 4 that guarantees feasibility and invariance of $\mathcal{X}_\chi^{\mathcal{F}}$ with respect to $\tilde{\xi}_{0|t} \in \Xi(\xi_{0|t})$ and $\Delta r \in \Omega_{\Delta r}(r(t))$. As a result, the proposed IAMPC solves Problem 1.

Result 1: For every initial condition in $\mathcal{X}_\chi^{\mathcal{F}}$, (2) in closed-loop with the IAMPC law (12), which solves (11) with prediction model (9), $P(\xi)$ satisfying (16), $\mathcal{C}_{xu}$ and $\mathcal{X}_\chi^f$ designed according to (19) and (20), respectively, and N selected according to Lemma 4, satisfies constraints (2c) and is ISS in $\delta = (\Delta x, e)$ with respect to the estimation error $\tilde{\xi}_{0|t}$ and the reference change $\Delta r(t)$. □

VI. Case Study

The case study is inspired by the real world application of compressor control in a variable refrigerant flow air conditioner (VRF-AC), for which a nominal MPC was designed in [12] based on a model obtained from first principles and experimental data, and a stabilizing IAMPC for a given and fixed setpoint was designed in [2]. Here we design the IAMPC that can track a changing reference for the room temperature, r_T[°C]. The system model is of 4^{th} order with the state coordinates being $x = [T_r\ T_e\ T_d\ \zeta]'$, where T_r[°C] is the room temperature, T_e[°C] is the evaporating temperature, T_d[°C] is the compressor discharge temperature, ζ is a nonphysical state related to internal conditions of the air conditioner, observable from the other states and that can be reconstructed by available measurements. The control input is the compressor frequency $u = C_f$[Hz]. The controller enforces upper and lower bounds on state, $\overline{x} = [26\ 14\ 70\ \infty]'$, $\underline{x} = [18\ 4\ 50\ -\infty]'$, and input $\overline{u} = 60$, $\underline{u} = 30$, where an infinite bound means that the variable is not (directly) constrained. Some of the bounds are tighter than in the actual application to better highlight the capability of the controller to enforce constraints. The uncertainty in the dynamics is due to the thermal mass of the room, here ranging within $\pm 35\%$ around the nominal value, and by the efficiency of the energy transfer from the evaporator to the room, here ranging within $\pm 20\%$ around the nominal value. The resulting MPC has $\ell = 4$ and $N = 12$ with $T_s = 1.5$min. We have implemented a simple parameter estimator that computes an unconstrained estimate of the parameter vector $\varrho(t)$ based on past data window of $N_m = 4$ steps, which is projected onto the simplex Ξ and filtered using the first order filter $\xi(t+1) = (1-\varsigma)\xi(t) + \varsigma \cdot \text{proj}_{\Xi}(\varrho(t)$, where $\varsigma \in (0,1)$, and $[\xi(0)]_i = 1/\ell$, $i \in \mathbb{Z}_{[1,\ell]}$. We have simulated the closed-loop 100 times, by selecting 10 initial conditions close (within 5%) to the border of the RCI set and simulating each for 10 plant realizations chosen randomly, with 50% probability of $\bar{\xi}$ being a random convex combination, and with 50% probability of $\bar{\xi}$ selecting one of the vertex systems.

The results are reported in Fig. 2 for a sequence of aggressive step changes, both up and down, in the room temperature reference. The constraints are enforced despite the uncertainty, and eventually offset free tracking is achieved. IAMPC only requires solving a QP, for which there are low complexity solvers suitable for execution in the microcontroller (here we use [13]), which points at the computational feasibility in the real application.

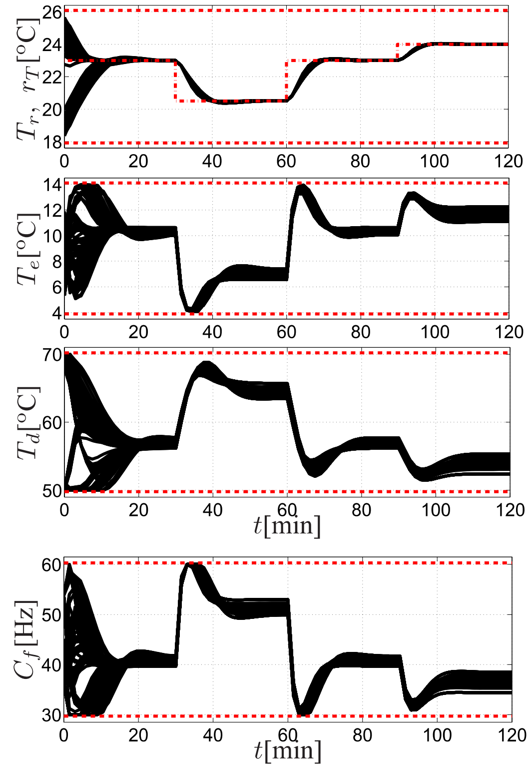

Fig. 2. Simulation results for the HVAC control case study. States T_r, T_e, T_d and input C_f (solid, black), room temperature reference r_T (dash-dot, red) and constraint bounds (dash, red).

VII. Conclusions

We proposed an output tracking indirect adaptive MPC for constrained linear systems with polytopic uncertainty. The method exploits the velocity form to achieve input-to-state stability for the closed-loop system, and robust constraint satisfaction and recursive feasibility are achieved via invariant sets. Simulation results for an application case study demonstrated the approach. IAMPC can be extended relatively easily towards multiple directions, for instance by allowing for additive disturbances, and allowing for concurrent reference manipulation as in [5], which will be subject of future works.

References

[1] D. Q. Mayne, "Model predictive control: Recent developments and future promise," *Automatica*, vol. 50, no. 12, pp. 2967–2986, 2014.

[2] S. Di Cairano, "Indirect adaptive model predictive control for liner systems with polytopic uncertainty," in *American Control Conf.*, 2016, pp. 3570–3575.

[3] D. Fan and S. Di Cairano, "Further results and properties of indirect adaptive model predictive control for linear systems with polytopic uncertainty," in *American Control Conf.*, 2016, pp. 2948–2953.

[4] R. Findeisen, L. Imsland, F. Allgower, and B. A. Foss, "State and output feedback nonlinear model predictive control: An overview," *European journal of control*, vol. 9, no. 2-3, pp. 190–206, 2003.

[5] D. Limón, I. Alvarado, T. Alamo, and E. F. Camacho, "Mpc for tracking piecewise constant references for constrained linear systems," *Automatica*, vol. 44, no. 9, pp. 2382–2387, 2008.

[6] G. Pannocchia and J. B. Rawlings, "The velocity algorithm LQR: a survey," Dep. Chemical Eng., Univ. Wisconsin, Madison, Tech. Rep. TWMCC-2001–004, 2001.

[7] G. Betti, M. Farina, and R. Scattolini, "A robust MPC algorithm for offset-free tracking of constant reference signals," *IEEE Transactions on Automatic Control*, vol. 58, no. 9, pp. 2394–2400, 2013.

[8] M. Huang, K. Zaseck, K. Butts, and I. Kolmanovsky, "Rate-based model predictive controller for diesel engine air path: Design and experimental evaluation," *IEEE Transactions on Control Systems Technology*, 2016, preprint available online.

[9] J. B. Rawlings and D. Q. Mayne, *Model predictive control: Theory and design*. Nob Hill Pub., 2009.

[10] J. Daafouz and J. Bernussou, "Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties," *Systems & control letters*, vol. 43, no. 5, pp. 355–359, 2001.

[11] F. Blanchini and S. Miani, *Set-theoretic methods in control*. Springer, 2008.

[12] N. Jain, D. J. Burns, S. Di Cairano, C. R. Laughman, and S. A. Bortoff, "Model predictive control of variable refrigerant flow systems," in *Int. Refrigeration and Air Conditioning Conf.*, 2014, available at http://docs.lib.purdue.edu/iracc/1545/.

[13] S. Di Cairano, M. Brand, and S. A. Bortoff, "Projection-free parallel quadratic programming for linear model predictive control," *Int. J. Control*, vol. 86, no. 8, pp. 1367–1385, 2013.