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Distributed Extremum Seeking in Multi-Agent Systems with Arbitrary Switching Graphs ⋆

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Abstract: This paper studies the problem of averaging-based extremum seeking in dynamical multi-agent systems with time-varying communication graphs. We consider a distributed consensus-optimization problem where the plants and the controllers of the agents share information via time-varying graphs, and where the cost function to be minimized corresponds to the summation of the individual response maps generated by the agents. Although the problem of averaging-based extremum seeking control in multi-agent dynamical systems with *time-invariant graphs* has been extensively studied, the case where the graph is time-varying remains unexplored. In this paper we address this problem by making use of recent results for generalized set-valued hybrid extremum seeking controllers, and the framework of switched differential inclusions and common Lyapunov functions. For the particular consensus-optimization problem considered in this paper, a semi-global practical stability result is established. A numerical example in the context of dynamic electricity markets illustrates the results.

Keywords: Extremum seeking, multi-agent systems, hybrid systems, adaptive control, optimization.

1. INTRODUCTION

Research on adaptive controllers has been motivated by the intrinsic uncertain and complex nature of real-world engineering plants deployed in unknown environments. One area where adaptive control design is recently very active, is in the area of multi-agent systems (MAS) control. One common characteristic in many MAS control problems is that they can be formulated as an optimization problem. This formulation allows us to use different tools from optimization and learning theory in order to solve the adaptive control problem in a distributed way, relying on local variables for each agent, and converging to a solution of the full optimization problem. A class of algorithms in adaptive control that has exploited this formulation is the so called black-box optimization control or extremum seeking control, see Ariyur and Krstic (2003), Tan et al. (2006), Poveda and Teel (2016), Benosman (2016). In the MAS setting an extremum seeking controller (ESC) is an algorithm that every agent implements aiming to optimize a measurable local or global performance indicator with unknown mathematical form, and subject to information constraints usually represented by a communication graph. Several works have addressed the continuous-time ESC problem in MAS, e.g., Kutadinata et al. (2015), Poveda and Quijano (2015), Menon and Baras (2014), Dougherty and Guay (2016), Ye and Hu (2016). With the notable exception of Dougherty and Guay (2016), most of these works are based on standard averaging and singular perturbation theory used for the analysis of smooth ESCs with periodic perturbations. Since in this approach the key stability properties of the controller are characterized by a reduced-average *time-invariant* system that usually corresponds to a perturbed version of a standard model-based distributed optimization algorithm, the study of MAS with arbitrarily *time-varying graphs* in extremum-seeking architectures based on averaging remains unexplored. We note that a hybrid-based approach for ESC in MAS with *slow* time-varying graphs satisfying a dwell-time constraint was presented in Poveda and Teel (2014). Since *arbitrarily* time-varying graphs may not satisfy a dwell-time constraint, said hybrid formulation cannot be applied for MAS with arbitrarily fast switching graphs.

Motivated by this background, and by recent stability and analysis tools for a class of *set-valued* ESCs presented in Poveda and Teel (2016), we present in this paper a set-valued extremum-seeking approach for MAS with *arbitrarily switching* graphs. Our results are based on the well-studied approach of modeling arbitrarily switching systems as a *time-invariant* differential inclusion that can generate any solution of the original switched system, provided the switching signal has finitely many discontinuities in every compact subinterval of its domain, see Liberzon

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and Morse (1999). The particular distributed optimization problem that we consider in this paper is similar to the one considered in Ye and Hu (2016), which corresponds to the case where a set of N agents with individual cost functions $J_i : \mathbb{R}^n \to \mathbb{R}$, aim to maximize a global function J defined as the summation of their individual cost. In the model-based setting, i.e., assuming perfect knowledge of J_i and ∇J_i, this problem has been extensively studied in the literature, e.g., Nedić and Ozdaglar (2009), Gharesifard and Cortés (2014), Lin et al. (2016), and in the ESC case with *time-invariant* communication graphs it was studied in Ye and Hu (2016). Nevertheless, we stress that the methodology presented in this paper for arbitrarily switching graphs can be easily extended and adapted to other type of distributed optimization problems in MAS where a common Lyapunov function exists.

The rest of this paper is organized as follows: In Section 2 we present some preliminary background and definitions for set-valued dynamical systems. In Section 3 we present the problem statement. Section 4 includes the main results. Section 5 presents a numerical example, and finally Section 6 ends with a conclusion.

2. PRELIMINARIES

The set of (nonnegative) real numbers is denoted by ($\mathbb{R}_{\geq 0}$) $\mathbb{R}$. The set of (nonnegative) integers is denoted by ($\mathbb{Z}_{\geq 0}$) $\mathbb{Z}$. We denote by $\mathbb{S}^1$ the unit circle in $\mathbb{R}^2$. We use $\mathbb{B}$ to denote a closed unit ball of appropriate dimension, $\rho\mathbb{B}$ to denote a closed ball of radius $\rho > 0$, and $\mathcal{X} + \rho\mathbb{B}$ to denote the union of all sets obtained by taking a closed ball of radius ρ around each point in the set $\mathcal{X}$. A set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is outer semi-continuous (OSC) at $x \in \mathbb{R}^m$ if for all sequences $x_i \to x$ and $y_i \in M(x_i)$ such that $y_i \to y$ we have that $y \in M(x)$. A set-valued mapping $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is locally bounded (LB) at $x \in \mathbb{R}^m$ if there exists a neighborhood U_x of x such that $M(U_x) \subset \mathbb{R}^n$ is bounded. Given a set $\mathcal{X} \subset \mathbb{R}^m$ the mapping M is said to be OSC and LB relative to $\mathcal{X}$ if the set-valued mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ defined by $M(x)$ for $x \in \mathcal{X}$ and $\emptyset$ for $x \notin \mathcal{X}$ is OSC and LB at each $x \in \mathcal{X}$. A set-valued mapping M is said to be locally Lipschitz in $\mathcal{X}$ if there exists $L \in \mathbb{R}_{\geq 0}$ such that $M(x) \subset M(y) + L|x - y|\mathbb{B}$ for all $x, y \in \mathcal{X}$. We use $\overline{\text{co}}\ \mathcal{X}$ to denote the closed convex hull of $\mathcal{X}$, $\overline{\mathcal{X}}$ to denote its closure, $\text{int}(\mathcal{X})$ to denote its interior, and $\mathcal{X}^N$ to denote the N-times Cartesian product of $\mathcal{X}$.

In this paper we work with general set-valued constrained differential inclusions of the form

$$\dot{x} \in F(x), \quad x \in C, \tag{1}$$

where the set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called the flow map and C is called the flow set. A solution of (1) is an absolutely continuous function $x : \text{dom}(x) \to \mathbb{R}^n$ that satisfies $\dot{x}(t) \in F(x(t))$ for almost all t in $\text{dom}(x)$, and $x(t) \in C$ for all $t \in \text{dom}(x)$. A solution x is *maximal* if there does not exist another solution ψ to (1) such that $\text{dom}(x)$ is a proper subset of $\text{dom}(\psi)$, and $x(t) = \psi(t)$ for all $t \in \text{dom}(x)$. A solution x is *complete* if $\text{dom}(x)$ is unbounded.

Definition 1. Let $\mathcal{A} \subset \mathbb{R}^n$ be a compact set. The set $\mathcal{A}$ is *uniformly globally asymptotically stable* (UGAS) for (1) if there exists a $\mathcal{KL}$ function β such that any maximal solution x of (1) satisfies $|x(t)|_{\mathcal{A}} \leq \beta(|x(0)|_{\mathcal{A}}, t)$, for all $t \in \text{dom}(x)$. □

Definition 2. For a differential inclusion of the form (1) parametrized by a vector of small positive parameters $\varepsilon := [\varepsilon_0, \ldots, \varepsilon_k]^\top$, a compact set $\mathcal{A} \subset \mathbb{R}^n$ is said to be semi-globally practically asymptotically stable (SGP-AS) as $(\varepsilon_0, \ldots, \varepsilon_k) \to 0^+$ if there exists a function $\beta \in \mathcal{KL}$ such that the following holds: For each $\Delta > 0$ and $\nu > 0$ there exists $\varepsilon_0^* > 0$ such that for each $\varepsilon_0 \in (0, \varepsilon_0^*)$ there exists $\varepsilon_1^*(\varepsilon_0) > 0$ such that for each $\varepsilon_1 \in (0, \varepsilon_1^*(\varepsilon_0)) \ldots$ there exists $\varepsilon_j^*(\varepsilon_{j-1}) > 0$ such that for each $\varepsilon_j \in (0, \varepsilon_j^*(\varepsilon_{j-1})) \ldots$, $j = \{2, \ldots, k\}$, each solution x that satisfies $|x(0)|_{\mathcal{A}} \leq \Delta$ also satisfies

$$|x(t)|_{\mathcal{A}} \leq \beta(|x(0)|_{\mathcal{A}}, t) + \nu, \tag{2}$$

for all $t \in \text{dom}(x)$. □

Note that Definitions 1 and 2 do not insist that each maximal solution x must have an unbounded time domain. However, Definition 1 asks that if the time domain of a particular solution is unbounded, then that solution must converge to $\mathcal{A}$. Similarly Definition 2 asks that every complete solution must converge to $\mathcal{A} + \nu\mathbb{B}$.

3. PROBLEM STATEMENT

Consider a network of $N \in \mathbb{Z}_{>0}$ dynamical systems, also called agents, characterized by an undirected graph $\mathcal{G}_{\mathcal{P}}(t) = (\mathcal{V}, \mathcal{E}_{\mathcal{P}}(t))$, where $\mathcal{V} := \{1, \ldots, N\}$ is the set of agents, and $\mathcal{E}_{\mathcal{P}}(t) \subset \mathcal{V} \times \mathcal{V}$ is the set of edges between agents. The graph $\mathcal{G}_{\mathcal{P}}(t)$ is assumed to satisfy the following assumption:

Assumption 3. For all $t \geq 0$ the undirected graph $\mathcal{G}_{\mathcal{P}}(t)$ is connected, with finitely many switches in every compact subinterval of $\mathbb{R}_{\geq 0}$. □

Since the number of agents is finite, for each $t \geq 0$ there are finitely many possible connected configurations that the graph $\mathcal{G}_{\mathcal{P}}(t)$ can take. Each of these configurations can be indexed by an integer state $q \in Q \subset \mathbb{Z}_{>0}$, where the cardinality of Q is the same as the number of connected configurations of $\mathcal{G}_{\mathcal{P}}(t)$. Based on this observation, in this paper we consider agents with plant dynamics $\mathcal{P}_i$ given by the differential inclusion

$$\dot{\theta}_i \in f_i^q\big(\theta, \alpha_i^q(\theta, u)\big), \quad \theta_i \in \Lambda_{\theta,i}, \quad y_i = \varphi_i^q(\theta, u_i), \tag{3}$$

which are indexed by the particular configuration $q \in Q$. In system (3) we have that:

- The vector $\theta_i \in \mathbb{R}^{p_i}$ is the state of the plant related to the i^{th} agent, $\theta = [\theta_1^\top, \ldots, \theta_N^\top]^\top \in \mathbb{R}^p$ is the overall plant state of the network, $\Lambda_{\theta,i} := \lambda_{\theta,i}\mathbb{B} \subset \mathbb{R}^p$, $\lambda_{\theta,i} > 0$ can be taken arbitrarily large to encompass any complete solution of practical interest, and $p := \sum_{i=1}^N p_i$.
- The state $u_i \in \mathbb{R}^n$ is the input of the i^{th} agent, and $u = [u_1, \ldots, u_N]^\top \in \mathbb{R}^{Nn}$ is the overall input to the network.
- For each $q \in Q$ the function $\alpha_i^q : \mathbb{R}^p \times \mathbb{R}^{nN} \to \mathbb{R}^n$ is a feedback law, pre-designed to stabilize the plant.
- For each $q \in Q$ the mapping $f_i^q : \mathbb{R}^p \times \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is in general set-valued.
- For each $q \in Q$ the function $\varphi_i^q : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}$ describes the output of the i^{th} agent.

- The mappings f_i^q, α_i^q, and φ_i^q are assumed to depend only on the states θ_j and inputs u_j of the neighboring agents of agent i, which are characterized by the set $\mathcal{N}_{\mathcal{P},i}(t) = \{j \in \mathcal{V} : (i,j) \in \mathcal{E}_{\mathcal{P}}(t)\}$.

Our motivation for considering set-valued mappings in (3), instead of standard Lipschitz differential equations, is that this allows us to also study plants described by ODEs with a discontinuous-right hand side, systems switching between a finite number of continuous vector fields, and plants that depend on unknown parameters that are known to belong to compact and convex sets.

Note that the graph $\mathcal{G}_{\mathcal{P}}(t)$ describes the interaction between the *physical* dynamics (3) of the agents of the network.

In a similar way, the control system $\mathcal{C}_i$ that regulates the input u_i of each agent $i \in \mathcal{V}$, i.e., the dynamics of u_i, is also allowed to share information only with a *time-varying* subset of the controllers $\mathcal{C}_j$ associated to the agents $j \in \mathcal{N}_{\mathcal{C},i}(t) := \{j : (i,j) \in \mathcal{E}_{\mathcal{C}}(t)\}$, where $\mathcal{E}_{\mathcal{C}}(t)$ is the set of edges of an undirected graph $\mathcal{G}_{\mathcal{C}}(t) := \{\mathcal{V}, \mathcal{E}_{\mathcal{C}}(t)\}$, which in general may be different from the plant's interaction graph $\mathcal{G}_{\mathcal{P}}(t)$. Note that the graph $\mathcal{G}_{\mathcal{C}}(t)$ describes the interaction between the *control* dynamics of the agents of the network, and since the number of controllers $\mathcal{C}_i$ is the same as the number of plants $\mathcal{P}_i$, each connected configuration of $\mathcal{G}_{\mathcal{C}}(t)$ can also be indexed by a logic state $c \in Q$.

The communication graph $\mathcal{G}_{\mathcal{C}}(t)$ of the control system must satisfy the following assumption:

Assumption 4. For all $t \geq 0$ the undirected graph $\mathcal{G}_{\mathcal{C}}(t)$ is connected, and it switches finitely many times in every compact subinterval of $\mathbb{R}_{\geq 0}$. □

In this paper it is assumed that the mappings $(f_i^q, \alpha_i^q, \varphi_i^q)$ are *unknown* for all configurations $q \in Q$, and only φ_i^q is accessible via measurements. However, the following regularity assumption is imposed on the dynamics (3):

Assumption 5. For each $q \in Q$ the set-valued mapping $(\theta, u) \rightrightarrows f_i^q(\theta, \alpha_i^q(\theta, u))$ is LB and locally Lipschitz, and $(\theta, u_i) \mapsto \varphi_i^q(\theta, u_i)$ is continuous. □

We also impose the following open-loop stability assumption on the overall networked system with individual dynamics (3).

Assumption 6. There exists a nonempty OSC and LB set-valued mapping $H : \mathbb{R}^{nN} \rightrightarrows \mathbb{R}^p$, such that for each $\rho \in \mathbb{R}_{>0}$ the networked system with individual agent dynamics (3) and *constant* input vector (i.e., $\dot{u} = 0$) restricted to the compact set $\mathbb{R}^{Nn} \cap \rho\mathbb{B}$, renders the set

$$\mathbb{M}_\rho := \{(\theta, u) : \theta \in H(u), u \in \mathbb{R}^{Nn} \cap \rho\mathbb{B}\} \tag{4}$$

UGAS for any time-varying graph $\mathcal{G}_{\mathcal{P}}(t)$ satisfying Assumption 3. □

Under Assumption 6, the set $\mathbb{M}_\rho$ must be UGAS for the networked switched system with dynamics (3) under any switching graph $\mathcal{G}_{\mathcal{P}}(t)$ that is connected for all $t \geq 0$.

Using the definition of $H(\cdot)$ in Assumption 6, the *response map* associated to each agent is defined as

$$J_i(u_i) := \{\varphi_i^q(\theta, u_i) : \theta \in H(u), q \in Q\}, \tag{5}$$

for all $i \in \mathcal{V}$. In order to have well-defined response maps $J_i(\cdot)$, we also impose the following additional regularity assumption.

Assumption 7. Let $H(\cdot)$ be given by Assumption 6. For each $i \in \mathcal{V}$, each pair $q, q' \in Q$, each $u \in \mathbb{R}^{nN}$, and each pair $\theta, \theta' \in H(u)$, we have that $\varphi_i^q(\theta, u_i) = \varphi_i^{q'}(\theta', u_i)$, and J_i in (5) only depends on u_i. □

With $J_i(\cdot)$ defined as in (5), and given the time-varying control's communication graph $\mathcal{G}_{\mathcal{P}}(t)$, the main goal of the agents is to cooperatively solve the following optimization problem in $\mathbb{R}^n$:

$$\text{minimize} \quad J(\cdot) := \sum_{i=1}^{N} J_i(\cdot), \tag{6}$$

without the knowledge of the mappings $f_i(\cdot,\cdot)$ and $\varphi_i(\cdot,\cdot)$ in (3) that generate $J_i(\cdot)$ in (5), and where each agent $i \in \mathcal{V}$ has access only to measurements of its own individual *output* φ_i of (3). In other words, agents aim to control their individual state u_i to agree on a common optimal point $u^* \in \mathbb{R}^n$ that minimizes (6). Since we seek to design gradient-based control mechanisms, the following smoothness, convexity, and boundedness assumption is imposed on the individual response-maps J_i.

Assumption 8. For each $i \in \mathcal{V}_i$ the response map J_i is analytic and convex. Moreover, the set of minimizers of J_i, given by $\mathcal{O}_i := \{u_i : \nabla J_i(u_i) = 0\}$, is nonempty and bounded, and $\nabla J_i(u_i)$ satisfies $\nabla J_i(u_i) = \rho_1 u_i + g_i(u_i)$, where $\|g_i(u_i)\| \leq \rho_2$ for all $u_i \in \mathbb{R}^n$, and $(\rho_1, \rho_2) \in \mathbb{R}^2_{>0}$ and $g_i(\cdot)$ are unknown. □

By Assumption 8 the set of minimizers of J in (6), defined as $\mathcal{O} := \{\tilde{u} : \nabla J(\tilde{u}) = 0\} \subset \mathbb{R}^n$, will be not empty and compact.

Before proceeding to the next section, let us discuss the implications and limitations imposed by Assumptions 3-8. In particular:

- The connectivity Assumptions 3 and 4 relax the standard fixed topology assumption considered in existing results of model-free optimization and extremum seeking, such as those in Kutadinata et al. (2015), Poveda and Quijano (2015), Menon and Baras (2014), Dougherty and Guay (2016), Ye and Hu (2016).
- Assumption 5 is trivially satisfied if the dynamics (3) are actually given by a differential equation with a continuous right hand-side.
- In most applications the graph $\mathcal{G}_{\mathcal{P}}(t)$ is time-invariant. In such case, Assumption 6 is just the generalization of the assumptions in Ariyur and Krstic (2003) and Nešić et al. (2010) for the case when the dynamics of the MAS are modeled by a differential inclusion rather than a differential equation.
- Assumption 7 is needed mainly to have a well-defined single-valued response map J. Additionally, if $H(\cdot)$ is actually a single-valued mapping, we recover the standard assumptions from Nešić et al. (2010).
- The definition of J assumes that its value is independent of the configuration $q \in Q$. Although this definition may seem too restrictive, it is worth noting that in most applications the output function φ_i^q of every agent is defined to be the same for any commu-

nication graph, which under Assumption 6, naturally leads to a q-independent response map J.

Finally, note that if the dynamics (3) of the agents are negligible such that the state θ can be omitted, the response map J_i in (5) is entirely defined by the output function $\varphi_i(u_i)$, and in this case one recovers the standard distributed optimization setting considered in Nedić and Ozdaglar (2009), Gharesifard and Cortés (2014), where the control's graph $\mathcal{G}_{\mathcal{C}}(t)$ characterizes the topology of the agents. Note, however, that here we do not assume perfect knowledge of the mathematical form of J_i and ∇J_i.

4. MAIN RESULT

To solve problem (6) the model-free feedback law u_i for each agent $i \in \mathcal{V}$ is defined as

$$u_i := \hat{u}_i + a \cdot \mathbb{D} \cdot \mu_i, \tag{7}$$

where $a \in (0, 1)$ is a tunable parameter, the matrix $\mathbb{D}$ is defined as

$$\mathbb{D} := [\mathbf{e}_1, \mathbf{0}_n, \mathbf{e}_2, \mathbf{0}_n, \ldots, \mathbf{e}_n, \mathbf{0}_n], \tag{8}$$

with $\mathbf{e}_i \in \mathbb{R}^n$ being a vector with the i^{th} entry equal to 1 and the rest equal to 0, and $\mathbf{0}_n$ is the zero vector in $\mathbb{R}^n$. The auxiliary states $\hat{u}_i \in \mathbb{R}^n$ and $\mu_i \in \mathbb{R}^{2n}$ are generated by the system

$$\dot{\hat{u}}_i \in k \cdot \hat{F}_{\delta,i}(\hat{u}_i, \hat{u}_j, \xi_i), \quad \forall\, j \in \mathcal{N}_{\mathcal{C},i}(t), \qquad \hat{u}_i \in \mathbb{R}^n \tag{9a}$$

$$\dot{\xi}_i = -\omega_L \cdot \left(\xi_i - 2a^{-1}\varphi_i(\theta, u_i) \cdot \mathbb{D} \cdot \mu_i\right), \quad \xi_i \in \Lambda_{\xi,i} \tag{9b}$$

$$\dot{\mu}_i = \Phi(\omega_i) \cdot \mu_i, \qquad \mu_i \in \mathbb{S}^n, \tag{9c}$$

where $\xi_i \in \mathbb{R}^n$ is also an auxiliary state evolving on the compact set $\Lambda_{\xi,i} := \lambda_i \mathbb{B}$, and the parameter $\lambda_i \in \mathbb{R}_{>0}$ is taken sufficiently large to encompass any complete solution of practical interest. The gains k and ω_L are defined as

$$k := \omega_L \cdot \sigma, \quad \omega_L := \epsilon \cdot \bar{\omega}, \quad (\sigma, \epsilon, \bar{\omega}) \in \mathbb{R}^3_{>0}. \tag{10}$$

The set-valued mapping $\hat{F}_{\delta,i}$ in Eq. (9a) is defined as

$$\hat{F}_{\delta,i}(\hat{u}, \xi_i) := \frac{1}{\delta} \cdot \sum_{j \in \mathcal{N}_{\mathcal{C},i}(t)} \overline{\text{sign}}(\hat{u}_j - \hat{u}_i) - \gamma \cdot \xi_i, \tag{11}$$

which is inspired on the model-based optimization dynamics presented in Lin et al. (2016), and where $\overline{\text{sign}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ maps each entry $z_i \in \mathbb{R}$ of a vector $z \in \mathbb{R}^n$ to a set $\overline{\text{sign}}(z_i) \subset \mathbb{R}$ defined as

$$\overline{\text{sign}}(z_i) := \begin{cases} \{1\}, & \text{if } z_i > 0 \\ [-1, 1], & \text{if } z_i = 0 \\ \{-1\}, & \text{if } z_i < 0 \end{cases}, \tag{12}$$

which corresponds to the Krasovskii regularization of the standard scalar function $\text{sign}(\cdot)$, i.e.,

$$\overline{\text{sign}}(z_i) := \bigcap_{\epsilon > 0} \overline{\text{co}}\ \text{sign}(z_i + \epsilon\mathbb{B}). \tag{13}$$

The matrix $\Phi(\omega_i) \in \mathbb{R}^{2n \times 2n}$ in Eq. (9c) is defined as

$$\Phi(\omega_i) := \begin{bmatrix} \Omega(\omega_{i,1}) & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \Omega(\omega_{i,2}) & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \Omega(\omega_{i,n}) \end{bmatrix}, \tag{14}$$

where the block components $\Omega(\omega_{i,j})$ are defined as

$$\Omega(\omega_{i,j}) := \begin{bmatrix} 0 & \omega_{i,j} \\ -\omega_{i,j} & 0 \end{bmatrix}, \quad \forall\, j \in \{1, \ldots, n\}, \tag{15}$$

with parameter $\omega_{i,j} = \epsilon \cdot \kappa_{i,j}$, where $\kappa_{i,j}$ is a rational number that satisfies $\kappa_{i,j} \neq \kappa_{i,k}$ for $j \neq k$ and $(j, k) \in \{1, \ldots, n\}$.

Under an appropriate time-scale separation between the learning dynamics (9) and the plant dynamics (3), which is achieved by selecting ϵ sufficiently small, the learning dynamics (9) can be qualitatively seen as a system comprised of three main components: a) an individual time-invariant signal generator given by Eq. (9c), used by every agent to generate an individual vector of dither signals with frequencies given by the parameters $\omega_{i,j}$ in (15), b) an individual "linear" gradient estimator given by Eq. (9b) which uses the matrix $\mathbb{D}$ to extract the components of μ_i that have different frequencies, and c) a distributed set-valued learning mechanism given by Eq. (9a), which uses estimates of ∇J_i, given by ξ_i, in order to distributively regulate the state $\hat{u}_i$ to a neighborhood of the compact set $\mathcal{O}$ that minimizes the function J via the mapping (11).

The combination of the learning dynamics (9), the feedback law (7), and the plant dynamics (3), lead to a networked system with overall state $x = [u^\top, \xi^\top, \mu^\top, \theta^\top]^\top \in \mathbb{R}^{4nN+p}$. For this closed-loop networked system we define the sets

$$\Lambda_\xi := \Lambda_{\xi,1} \times \ldots \times \Lambda_{\xi,N}, \quad \Lambda_\theta := \Lambda_{\theta,1} \times \ldots \times \Lambda_{\theta,N}. \tag{16}$$

The following theorem characterizes the stability properties of the closed-loop networked system.

Theorem 9. Suppose that Assumptions 3-8 hold, and that every agent $i \in \mathcal{V}$ with dynamics (3) implements the model-free optimization dynamics (9) with $\gamma > 0$. Then, for each compact set $K \subset \mathbb{R}^n$ satisfying $\mathcal{O} \subset \text{int}(K)$, and for each agent $i \in \mathcal{V}$, there exists $(\lambda_{\xi,i}, \lambda_{\theta,i}) \in \mathbb{R}^2_{>0}$ such that the set $\mathcal{O}^N \times \Lambda_\xi \times (\mathbb{S}^n)^N \times \Lambda_\theta$ is GP-AS as $(\delta, a, \sigma, \bar{\omega}, \epsilon) \to 0^+$ for the closed-loop networked system with flow set

$$C := (\mathbb{R}^n \cap K)^N \times \Lambda_\xi \times (\mathbb{S}^n)^N \times \Lambda_\theta. \tag{17}$$

Proof. The proof is presented in the Appendix.

Since the compact set K in Theorem 9 can be selected arbitrarily large, the stability result is of semi-global practical nature. Namely, for each $K \subset \mathbb{R}^n$ containing $\mathcal{O}$, and each $\nu > 0$, the parameters $(\delta, a, \sigma, \bar{\omega}, \epsilon)$ can be selected sufficiently small such that there will exist a time $T > 0$ such that complete solutions starting in K will satisfy $u_i(t) \in \mathcal{O} + \nu\mathbb{B}$ for all $t \geq T$ and all $i \in \mathcal{V}$.

Remark 10. Note that the result of Theorem 9 does not insist that every solution of the closed-loop system with flow set will be complete. However, by compactness of K every solution will be bounded, and the u_i component of every *complete* solution must converge to a neighborhood of $\mathcal{O}$. Moreover, the existence of complete solutions is guaranteed, and every complete solution of practical interest can be obtained by selecting the set K, and the constants $\lambda_{\theta,i}$ and $\lambda_{\xi,i}$ sufficiently large.

The methodology used to prove Theorem 9, presented in Appendix A, is based on analyzing the time-varying systems (3) and (9a) as *time-invariant* differential inclusions defined as the closure of the convex hull of the union of all possible vector fields obtained by the connected configurations of their respective graphs. This methodology is not restricted to the problem under consideration here, and it can be applied to most of the existing distributed ESCs for MAS in the literature, provided a common Lyapunov function exists for the optimization algorithm.

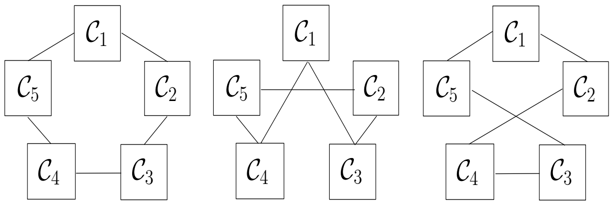

Fig. 1. Three communication graphs for the users of the network.

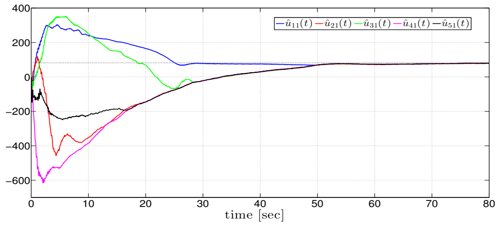

Fig. 2. Evolution in time of $\hat{u}_1$ for each agent $i \in \mathcal{V}$.

5. NUMERICAL EXAMPLE

We present a simple application in a networked system with 5 nodes, similar to the one considered in (Ye and Hu, 2016, Section VII), but relaxing the time-invariant connectivity of the graph. In particular, we consider an electricity market with 5 users equipped with individual loads. Each user has an individual model-free control/optimization system that shares information with other users via a time-varying undirected graph that can take any of the three configurations shown in Figure 1. For simplicity we consider that the users have no internal physical dynamics associated to a state θ, such that the response maps J_i are entirely given by the output functions φ_i, and defined as

$$J_i(u_i) := \rho_i(u_{ii} - \bar{u}_{ii})^2 + u_i \left(\frac{1}{2} \left(\sum_{j=1}^{n} u_{ij} - 640 \right) + 10 \right), \tag{18}$$

which is only accessible to the i^{th} agent via measurements, and where $\bar{u} = [120, 140, 160, 180, 200]^\top$, and $\rho_i = [5.2,\ 5.4,\ 5.6,\ 5.8,\ 6]^\top$. For a justification of the structure and parameter values of the cost functions (18) see Ma et al. (2014). By using the communication graph $\mathcal{G}_\mathcal{C}(t)$, and measuring their individual cost function (18), the agents aim to cooperatively find the optimal energy consumption $u^* \in \mathbb{R}^n$ that minimizes the social cost $J = \sum_{i=1}^{N} J_i$, i.e., agents aim to agree in a common optimal point u^* such that $u_i^* = u_{1i} = u_{2i} = \ldots = u_{Ni}$, for all $i \in \{1, \ldots, n\}$. The theoretical optimal solution to the consensus-optimization problem (18) is given by $u^* = [82.23,\ 103.63,\ 124.93,\ 146.14,\ 167.27]^\top$. Figures 2-6 show the evolution in time of each component j of the state $\hat{u}_{ij}$ of each agent i, where it can be seen that the estimate of each agent of the optimal value of u_i^* converges to a neighborhood of u_i^*. For this simulation the parameters used were $a = 0.2$, $\omega = [30, 25, 35, 40, 37.5]^\top$, $\omega_L = 1$, $\delta = 0.1$, $\gamma = 0.1$, and $k = 1.2$. The graph $\mathcal{G}_\mathcal{C}(t)$ switches every 0.1 seconds between the three configurations shown in Figure 1.

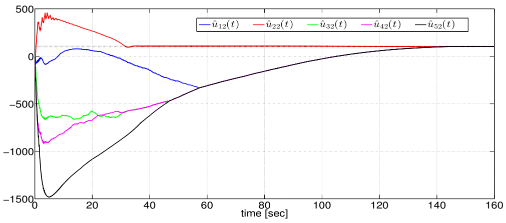

Fig. 3. Evolution in time of $\hat{u}_2$ for each agent $i \in \mathcal{V}$.

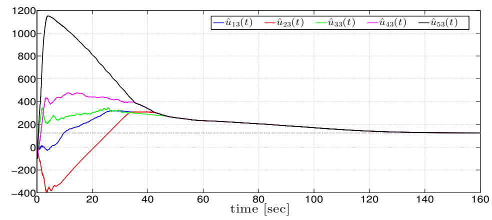

Fig. 4. Evolution in time of $\hat{u}_3$ for each agent $i \in \mathcal{V}$.

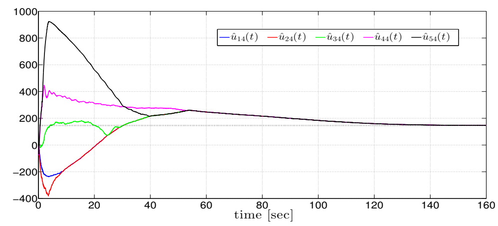

Fig. 5. Evolution in time of $\hat{u}_4$ for each agent $i \in \mathcal{V}$.

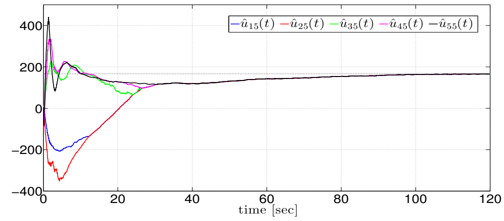

Fig. 6. Evolution in time of $\hat{u}_5$ for each agent $i \in \mathcal{V}$.

6. CONCLUSION

This paper presented an averaging-based ESC for MAS with arbitrarily fast switching graphs. The analysis of the method is based on formulating the switching optimization dynamics as a differential inclusion constructed as the closure of the convex hull of the union of all vector fields obtaining by each connected configuration of the graph. The methodology used in this paper can be easily applied and extended to other type of distributed optimization problems with time-varying graphs where a common Lyapunov function exists. The results were illustrated with a numerical example in the context of electricity markets with users equipped with individual loads.

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Appendix A. PROOF OF THEOREM 9

Since the networked system has arbitrarily switching graphs $\mathcal{G}_{\mathcal{P}}(t)$ and $\mathcal{G}_{\mathcal{C}}(t)$ satisfying Assumptions 3-4, the behavior of the closed-loop system can be studied by formulating its dynamics as a non-switched differential inclusion. Indeed, since q and c are indexing the connected configurations of $\mathcal{G}_{\mathcal{P}}(t)$ and $\mathcal{G}_{\mathcal{C}}(t)$, every solution of the networked system with switching plants (3) and fixed input $u \in \mathbb{R}^{nN} \cap \rho\mathbb{B}$ would also be a solution of the time-invariant constrained differential inclusion (in vectorial form)

$$\dot{\theta} \in f(\theta, u) := \overline{\text{co}} \bigcup_{q \in Q} f^q(\theta, \alpha^q(\theta, u)), \quad \dot{u} = 0, \tag{A.1a}$$

$$C_\theta := \mathbb{R}^p \times (\mathbb{R}^{nN} \cap \rho\mathbb{B}), \quad \rho > 0. \tag{A.1b}$$

Since the switched system $f^{q(t)}$ renders the set (4) UGAS under any switching signal $q : \mathbb{R}_{\geq 0} \to Q$, the following lemma follows by the results in Mancilla-Aguilar et al. (2005a) and Mancilla-Aguilar et al. (2005b).

Lemma 11. Suppose that Assumptions 3-6 hold. Then $f(\cdot, \cdot)$ is OSC and LB, and for each $\rho > 0$ system (A.1a) with flow-set (A.1b) renders the set (4) UGAS. □

By a similar argument, solutions of the learning dynamics (9a), in vectorial form, with switching logic mode $c : \mathbb{R}_{\geq 0} \to Q$ satisfying Assumption 3, are also solutions of the time-invariant differential inclusion

$$\dot{\hat{u}} \in \hat{F}_\delta(\hat{u}, \xi) := \overline{\text{co}} \bigcup_{c \in Q} \hat{F}^c_\delta(\hat{u}, \xi), \quad \hat{u} \in \mathbb{R}^{nN}, \tag{A.2}$$

where $\hat{F}^c_\delta := \hat{F}^c_{\delta,1} \times \ldots \times \hat{F}^c_{\delta,N}$, $\hat{u} = [\hat{u}_1, \ldots, \hat{u}_N]^\top$, and $\xi := [\xi_1, \ldots, \xi_N]^\top$. For the case when $\xi_i = \nabla J_i(u_i)$, the stability properties of (A.2) are characterized by the following lemma which is a direct consequence of the common Lyapunov function used in the proof of (Lin et al., 2016, Thm. 1).

Lemma 12. Consider the learning dynamics (A.2) with set-valued mapping (11), $\gamma > 0$, $\xi_i := \nabla J_i(u_i)$, and flow set $C_u = \mathbb{R}^{Nn}$. Then, there exists a $\delta > 0$ such that the set $\mathcal{O}^N$ is UGAS for (A.2). □

Finally, using the formulations (A.1a) and (A.2) we obtain the complete closed-loop time-invariant system given by

$$C := \mathbb{R}^{nN} \times \Lambda_\xi \times (\mathbb{S}^n)^N \times \Lambda_\theta, \tag{A.3a}$$

$$\dot{x} \in F(x) := \begin{pmatrix} k \cdot \hat{F}_\delta(\hat{u}, \xi) \\ -\omega_L \cdot \left(\xi - \dfrac{2}{a} \varphi(\theta, u) \cdot [I_{N \times N} \otimes \mathbb{D}] \cdot \mu \right) \\ \Phi(\omega) \cdot \mu \\ f(\theta,\ \hat{u} + a\ \mathbb{D} \cdot \mu) \end{pmatrix}, \tag{A.3b}$$

where $\otimes$ is the Kronecker product, $\Phi(\omega)$ is defined as

$$\Phi(\omega) := \begin{bmatrix} \Phi(\omega_1) & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \Phi(\omega_2) & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \Phi(\omega_N) \end{bmatrix}, \tag{A.4}$$

$\varphi := [\varphi_1(\theta, u_1), \ldots, \varphi_N(\theta, u_N)]^\top$, and $\omega = [\omega_1, \ldots, \omega_N]^\top$. System (A.3) is a type of set-valued hybrid extremum seeking control with empty jump set and jump map, studied in Poveda and Teel (2016). Using Lemmas 11-12, and the completness of solutions of system (A.2), the stability result follows by (Poveda and Teel, 2016, Thm.1).