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Irregular Polar Coding for Massive MIMO Channels

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***Abstract*—Recent advancements in polar coding have achieved practical performance competitive with other capacity-achieving codes such as low-density parity-check codes. In this paper, we propose a new family called irregular polar codes, where polarlization units are irregularly inactivated to achieve additional degrees of freedom for code design. We first discuss the code construction for irregular polar-coded modulation by taking non-uniform bit-reliability into consideration. We then apply the proposed polar codes to wireless massive multiple-input multiple-output (MIMO) communication channels. Simulation results show that the irregular polar codes can significantly reduce encoding/decoding complexity up to 50% while also yielding a marginal improvement in error rate performance.**

I. Introduction

Recently, polar codes [1] have been regarded as strong candidates for the fifth-generation (5G) wireless communications systems [2]. They are able to provide low error rate performance [2–16], which is highly competitive against other capacity-approaching codes such as low-density parity-check (LDPC) codes. Therefore, it is of interest to study polar codes for the 5G wireless communications, particularly in the context of massive multiple-input multiple-output (MIMO) systems. For 5G networks, internet-of-things (IoT) [17] have received much attention as an important application to realize seamless inter-connections among a great number of heterogeneous systems, such as wearable devices, smart phones and augmented reality devices. Such systems are often energy and latency constrained, and thus require relatively short-length codes with low encoding/decoding complexity.

Arıkan has shown that low-complexity successive cancellation (SC) for polar decoding can achieve capacity for arbitrary discrete-input memoryless channels (DMC) [1]. More recently, it was reported that SC list decoding with cyclic redundancy check (CRC) outperforms state-of-the-art LDPC codes [9–11]. For energy-efficient applications such as IoT, a number of lower-complexity variants of polar list decoding algorithms have also been proposed [12–16].

In this paper, we propose a new family of polar codes, whose polarization units are irregularly pruned to drastically reduce the computational complexity in both encoding and decoding. In the context of LDPC codes, it is well-known that irregular codes with specific degree distributions outperform regular ones. Relaxed polar codes [18] were proposed to incorporate irregularity, where some polarization units are inactivated when the bit-channels are sufficiently good or bad. However, the relaxed polar code [18] does not consider all possible combinations of inactivation; specifically once a polarization unit is inactivated, further polarizations after the inactivated unit are not considered. This limits the flexibility in selection of polarization units that can be inactivated and does not exploit the possibility of further complexity reduction. In this paper, we propose fully irregular polar codes by taking the non-uniform reliability of bit-channels into consideration when performing inactivation of polarization units. We show that the performance can be improved compared to the conventional regular polar codes, while also significantly reducing encoding and decoding complexity up to 50%.

The contribution of this paper is summarized as follows.

- **Joint design of frozen bit and interleaver**: We first introduce the use of extrinsic information transfer (EXIT) analysis [19] to jointly optimize the frozen bit locations and interleaver, by exploiting non-uniform reliability in different bit-planes of high-order modulations.
- **Irregular inactivation of polarization units**: We generalize the idea in [18] to fully consider all possible selection of the inactivated polarization units for significant reduction of the encoding/decoding complexity.
- **Efficient code construction algorithm**: We propose a greedy algorithm to construct irregular polar codes that have the lowest bit error rate (BER) given particular channels and modulation schemes.
- **One-shot massive MIMO transmission**: We apply irregular polar coding to massive MIMO transmission in wireless fading channels. To realize the minimum latency and maximum antenna diversity, a whole polar codeword is multiplexed at once through massive antennas.

II. System Description

A. Polar Codes

Polar codes [1] have the ability to achieve the capacity over arbitrary DMCs, by leveraging the channel polarization phenomenon. An (N, k) polar code with k information bits and N encoded bits ($N = 2^n$) uses an $N \times N$ generator matrix $\boldsymbol{F}^{\otimes n}$ for encoding, where $\boldsymbol{F} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is a binary kernel and $[\cdot]^{\otimes n}$ denotes the n-fold Kronecker power. Let $\boldsymbol{u} = [u_1, u_2, \ldots, u_N]^{\mathrm{T}}$ and $\boldsymbol{x} = [x_1, x_2, \ldots, x_N]^{\mathrm{T}}$ respectively denote the vectors of input bits and encoded bits where $\boldsymbol{x} = \boldsymbol{F}\boldsymbol{u}$, and $\boldsymbol{y} = [y_1, y_2, \ldots, y_N]^{\mathrm{T}}$ denotes the vector of

C. Cao conducted this research while he was an intern at MERL.

decoder inputs. Due to the assumption of memorylessness, the transition probability $W_N(\boldsymbol{y}|\boldsymbol{x})$ between $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as $W_N(\boldsymbol{y}|\boldsymbol{x}) = \prod_{i=1}^{N} W(y_i|x_i)$.

The polar coding maps the information bits to the k most reliable locations in $\boldsymbol{u}$. The remaining $N-k$ input bits are "frozen" bits, fixed to values known to both the encoder and the decoder. We use $\mathbb{K}$ and $\bar{\mathbb{K}}$ to denote the subsets of $\{1,2,\ldots,N\}$ that correspond to the information bit and frozen bit locations, respectively. By using various methods [1, 20–22] including density evolution (DE), the locations in $\boldsymbol{u}$ with the lowest reliability can be selected as $\bar{\mathbb{K}}$. Due to the nature of Kronecker product, polar encoding and decoding can be performed at a complexity on the order of $\mathcal{O}(N\log_2 N)$.

B. Polar-Coded High-Order Modulation

In order to achieve higher spectral efficiency, we consider higher-order modulation schemes whose constellation $\mathcal{A}$ has $|\mathcal{A}| = 2^m$ signal points. The modulation index l is determined by the length-m binary tuple $[x_1, x_2, \ldots, x_m]$. For multi-layer coding (MLC) [23–25], m component codes are used which have the same length L_c but different rates. A modulated symbol is formed by mapping each coded bit in the m component codes to a different bit-plane of the constellation point. An upper bound on the achievable rate of the MLC schemes is obtained using the chain rule as follows [23]:

$$C_\mathrm{MLC} = \frac{1}{m}\sum_{i=1}^{m} I(\boldsymbol{y}; x_i|x_{i-1},\ldots,x_1) = \frac{1}{m}\sum_{i=1}^{m} C_i, \quad (1)$$

where $I(\boldsymbol{y}; x_i|x_{i-1},\ldots,x_1)$ denotes the mutual information of the ith bit conditioned on the lower significant bit-planes.

The MLC decoder of the ith component code uses estimates of $[\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{i-1}]$ that result from decodings of the bit-planes of lower significance. A drawback of such multistage decoding is its long latency. Decoding the ith component code must be delayed until the decoding has completed for all codes of lower significance. Furthermore, one is required to design multiple component codes, whose code rates must match the conditional mutual information C_i individually.

As a more practical approach than MLC, bit-interleaved coded-modulation (BICM) [7–9] uses a single code whose rate agrees with the averaged mutual information calculated over symbols of differing significance. The interleaver is used to randomize the non-identical reliability. The achievable rate of BICM schemes is bounded as

$$C_\mathrm{BICM} = \frac{1}{m}\sum_{i=1}^{m} I(\boldsymbol{y}; x_i) = \frac{1}{m}\sum_{i=1}^{m} \widetilde{C}_i \le C_\mathrm{MLC}, \quad (2)$$

where $\widetilde{C}_i := I(\boldsymbol{y}; x_i)$ is the unconstrained mutual information for the ith bit-plane.

Although the upper bound on BICM cannot exceed that on MLC in theory, in practice BICM schemes have the potential to achieve better performance than MLC, as discussed in [24]. This is because in BICM the codeword length of the underlying code can be m-times longer than any m component codes used in MLC. The effect can be significant at short block

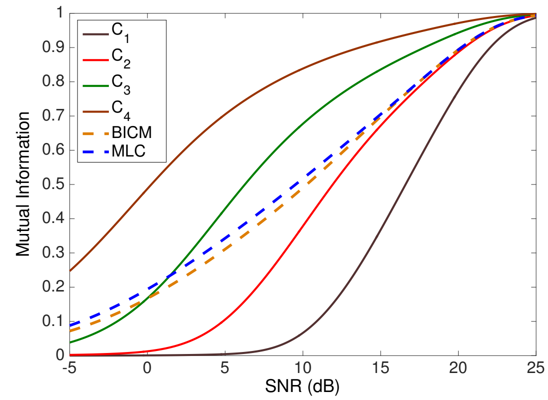


Fig. 1. MLC and BICM bounds for 256QAM (16PAM in each quadrature).

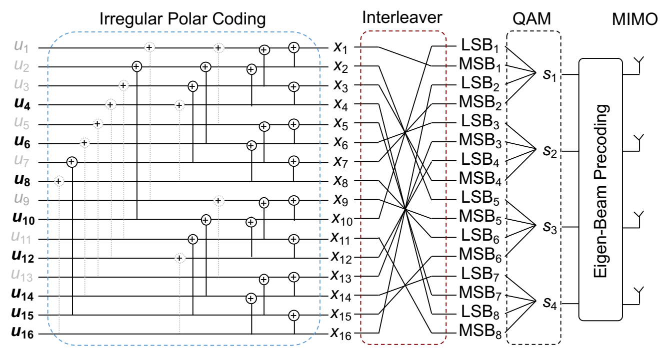


Fig. 2. Irregular polar-coded BICM scheme for massive MIMO transmission.

lengths and higher-order modulations. The mutual information of BICM systems using 256-ary quadrature-amplitude modulation (256QAM) is plotted in Fig. 1 for different bit-planes. The figure also compares the mutual information (averaged by the number of bits m in symbol) of BICM with MLC. It can be observed that the gap between MLC and BICM bounds is negligible with higher code rate. It can also be observed that the reliability of each bit-plane is not identical, which yields non-uniformity in bit reliability that can be exploited in the design of polar codes for BICM systems. Specifically, the DMC transition probabilities $W_i(y_i|x_i)$ are no longer identical across bit locations i.

C. Single-User Massive MIMO Transmission

Fig. 2 illustrates the system under consideration, where the polar-coded bits $\boldsymbol{x}$ are first interleaved via a permutation denoted by $\mathit{\Pi}(\boldsymbol{x})$ followed by the 2^m-ary QAM modulator to generate $N' = N/m$ symbols $\boldsymbol{s} = [s_1, s_2, \ldots, s_{N'}]^\mathrm{T}$. The modulated symbols are multiplexed in eigen-modes by N' transmitter antennas to have the minimum latency. The receiver having N' antenna branches obtains the received signal $\boldsymbol{r} = [r_1, r_2, \ldots, r_{N'}]^\mathrm{T}$, which is modeled as $\boldsymbol{r} = \boldsymbol{HDs} + \boldsymbol{z}$ where the $N' \times N'$ channel matrix can be expressed as $\boldsymbol{H} = \boldsymbol{SVD}^\mathrm{H}$ according to the singular value decomposition, $\boldsymbol{D}$ is the eigen-mode precoding matrix, and

$\boldsymbol{z}$ is a complex additive white Gaussian noise (AWGN) vector with an element-wise variance of $1/\rho$. For simplicity, the MIMO channel coefficient $H_{i,j}$ with $\mathbb{E}[|H_{i,j}|^2] = 1$ follows an independent and identically distributed (*i.i.d.*) Rayleigh fading. After demodulation and de-interleaving, the receiver provides log-likelihood ratio (LLR) data as input to the polar decoder, which employs SC list decoding with CRC [10].

III. Irregular Polar Codes

A. Basic Concept of Irregular Polarization

In this section, we explain the basic idea of irregular polar codes, which is to inactivate some polarization units to obtain potential error-rate performance improvement and complexity reduction. We first consider the conventional Bhattacharyya parameter analysis [1] to discuss the irregular polar coding, and will later propose a more sophisticated code construction method using EXIT [19] in the next section.

For an (N, k) polar coding, there are $n = \log_2(N)$ polarization stages, each of which contains $N/2$ polarization units (thus $N_{\mathrm{U}} \triangleq N \log_2(N)/2$ polarization units in total), as shown in Fig. 3(a) for $N = 4$. We propose a new family called irregular polar coding where some of the polarization units are inactivated, as exemplified in Figs. 3(b) through (d). We denote $\mathcal{U}(r, l)$ as the rth polarization unit from top to bottom in the lth polarization stage from left to right. For example, irregular polar codes in Figs. 3(b), (c), and (d) have an inactivated polarization unit at $\mathcal{U}(1, 1)$, $\mathcal{U}(2, 1)$, and $\mathcal{U}(1, 2)$, respectively. At the inactivated polarization unit, the exclusive-or (XOR) operation is removed.

B. Bhattacharyya Parameter

The word error rate (WER) of the irregular polar codes for the binary erasure channel (BEC) can be expressed via Bhattacharyya parameter evolution [1]. Starting from the channel side (nth polarization stage), the Bhattacharyya parameter $Z_i^{[n]}$ is given as

$$Z_i^{[n]} \triangleq \sum_{y_i \in \mathcal{Y}} \sqrt{W_i(y_i|0) W_i(y_i|1)} = \epsilon. \tag{3}$$

where ϵ is the erasure probability of the BEC. Let $Z_{r_{\mathrm{U}}}^{[l]}$ and $Z_{r_{\mathrm{L}}}^{[l]}$ be the incoming Bhattacharyya parameters, respectively, for upper and lower branches at polarization unit of $\mathcal{U}(r, l)$. If a polarization unit $\mathcal{U}(r, l)$ is inactivated, the bit reliabilities do not change as follows:

$$Z_{r_{\mathrm{U}}}^{[l-1]} = Z_{r_{\mathrm{U}}}^{[l]}, \qquad Z_{r_{\mathrm{L}}}^{[l-1]} = Z_{r_{\mathrm{L}}}^{[l]}. \tag{4}$$

Whereas, the evolution of the Bhattacharyya parameters is given for active polarization units as follows:

$$Z_{r_{\mathrm{U}}}^{[l-1]} = Z_{r_{\mathrm{U}}}^{[l]} + Z_{r_{\mathrm{L}}}^{[l]} - Z_{r_{\mathrm{U}}}^{[l]} Z_{r_{\mathrm{L}}}^{[l]}, \qquad Z_{r_{\mathrm{L}}}^{[l-1]} = Z_{r_{\mathrm{U}}}^{[l]} Z_{r_{\mathrm{L}}}^{[l]}. \tag{5}$$

Note that the above evolution will be identical to the original polar codes [1] only if the incoming Bhattacharyya parameters at upper and lower branches of polarization units are equal, $Z_{r_{\mathrm{U}}}^{[l]} = Z_{r_{\mathrm{L}}}^{[l]}$. In order to account for possible non-uniformity of bit reliabilities, we shall use the modified evolution in (5).

In fact, non-uniform bit reliabilities occur in various situations, e.g., when polar coding is used for high-order modulation schemes [7, 8] as discussed in Fig. 1, and/or in frequency-selective fading channels [9]. Moreover, our proposed polar codes with irregular inactivation can inherently produce non-uniform reliabilities even when the channels are identical. This is illustrated in Fig. 3(d), where an inactivated polarization unit at $\mathcal{U}(1, 2)$ involves non-equal Bhattacharyya parameters for the first polarization stage $\mathcal{U}(1, 1)$ and $\mathcal{U}(2, 1)$.

Once we obtained the left-most Bhattacharyya parameters $Z_i^{[0]}$, the WER P_ϵ of SC decoding is expressed as follows [1]:

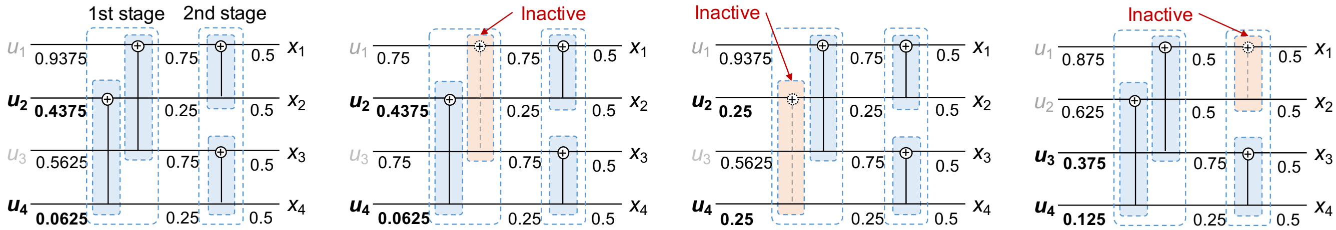
$$P_\epsilon = 1 - \prod_{i \in \mathbb{K}} \left(1 - \frac{Z_i^{[0]}}{2}\right) < \sum_{i \in \mathbb{K}} \frac{Z_i^{[0]}}{2}. \tag{6}$$

In Fig. 3, we present the WER and the Bhattacharyya parameters at every polarization unit for a channel erasure rate of $\epsilon = 0.5$. The WER of the conventional regular polar code in Fig. 3(a) is $P_\epsilon = 0.2365$, whereas the irregular polar codes in Figs. 3(b), (c), and (d) perform no worse than the regular polar code; i.e., $P_\epsilon = 0.2365$, 0.21875, and 0.22656, respectively. This indicates that with appropriate inactivation of polarization units, irregular polar codes can outperform regular polar codes. More importantly, since no computation is required for the inactivated polarization units during encoding and decoding, the computational complexity of irregular polar codes can be significantly reduced without any penalty.

C. Motivations

The above-mentioned benefits were partly discussed in relaxed polar coding [18], which inactivates a series of polarization units when the incoming Bhattacharyya parameters are sufficiently good or bad. For example, the polarization unit at $\mathcal{U}(1, 1)$ in Fig. 3(b) has already poor messages ($Z_{1_{\mathrm{U}}}^{[1]} = Z_{1_{\mathrm{L}}}^{[1]} = 0.75$), and thus no performance degradation is incurred by inactivating it because u_1 and u_3 will be chosen as frozen bits. Another example in Fig. 3(c) inactivated the polarization unit $\mathcal{U}(2, 1)$ since the second polarization stage already created good messages ($Z_{2_{\mathrm{U}}}^{[1]} = Z_{2_{\mathrm{L}}}^{[1]} = 0.25$). This irregular code achieves not only reduced complexity but also improved WER because the worst information bit u_2 is improved from 0.4375 to 0.25 by flattening reliability compared to the regular one.

However, relaxed polar coding [18] has limited flexibility in the inactivation pattern in order to accommodate conventional code construction methods [1] while preventing non-uniform reliabilities. For example, relaxed polar coding excludes the case in Fig. 3(d), where the first polarization units $\mathcal{U}(1, 1)$ and $\mathcal{U}(2, 1)$ are still active even after inactivating the polarization unit at $\mathcal{U}(1, 2)$. In this paper, we generalize the concept in [18] by considering arbitrary irregular inactivation with the help of a modified code construction method, which takes non-uniform reliability into consideration, in order to further reduce the computational complexity.



(a) Regular: $P_\epsilon = 0.2365$ (b) Irregular $\mathcal{U}(1,1)$: $P_\epsilon = 0.2365$ (c) Irregular $\mathcal{U}(2,1)$: $P_\epsilon = 0.21875$ (d) Irregular $\mathcal{U}(1,2)$: $P_\epsilon = 0.22656$

Fig. 3. Examples of inactivated polarization units for $(4,2)$ irregular polar coding (corresponding WER is P_ϵ).

IV. Design Method for Irregular Polar Coding

A. EXIT Evolution with Non-Uniformity

Now we consider generalized channels including massive MIMO channels instead of BEC to design fully irregular polar coding, where there are a maximum of $N_\mathrm{U} = N\log_2(N)/2$ polarization units that can be inactivated. We address how to accommodate the non-uniform bit-LLRs to construct irregular polar codes by appropriately inactivating polarization units that lead to the best performance and minimum complexity.

As discussed in the previous section, the conventional construction methods [20–22] assuming uniform bit-LLRs cannot be applied to optimize polar codes in the presence of irregular inactivation (and/or fading, high-order modulations, etc.). Motivated by a fact that polar codes can be represented by a factor graph [1], we propose to introduce EXIT analysis [19] for tracking the non-uniform mutual information at every polarization unit. Letting $\mathcal{I}_{r\mathrm{U}}^{[l]}$ and $\mathcal{I}_{r\mathrm{L}}^{[l]}$ denote mutual information of the upper and lower branches at the rth active polarization unit of lth stage, we use the following evolution:

$$\begin{aligned}\mathcal{I}_{r\mathrm{U}}^{[l-1]} &= 1 - J\Big(\sqrt{\big[J^{-1}(1-\mathcal{I}_{r\mathrm{U}}^{[l]})\big]^2 + \big[J^{-1}(1-\mathcal{I}_{r\mathrm{L}}^{[l]})\big]^2}\Big),\\ \mathcal{I}_{r\mathrm{L}}^{[l-1]} &= J\Big(\sqrt{\big[J^{-1}(\mathcal{I}_{r\mathrm{U}}^{[l]})\big]^2 + \big[J^{-1}(\mathcal{I}_{r\mathrm{L}}^{[l]})\big]^2}\Big),\end{aligned} \tag{7}$$

where $J(\cdot)$ is ten Brink's J-function [19], defined as

$$J(x) = 1 - \int_{-\infty}^{\infty} \frac{e^{-(t-x^2/2)^2/2x^2}}{\sqrt{2\pi x^2}} \log_2(1+e^{-t})\,\mathrm{d}t, \tag{8}$$

and $J^{-1}(\cdot)$ is its inverse function. For inactive polarization units, mutual information is propagated without modification.

We perform the EXIT evolution in (7) iteratively to trace the mutual information of each stage until we obtain the output mutual information $\mathcal{I}_i^{[0]}$ for all $i \in \{1, 2, \ldots, N\}$. Once the output mutual information $\mathcal{I}_i^{[0]}$ are obtained, the union bound of error rate P_e is derived as below:

$$P_\mathrm{e} = \frac{1}{|\mathbb{K}|}\sum_{i\in\mathbb{K}} Q\Big(\frac{1}{2}J^{-1}(\mathcal{I}_i^{[0]})\Big), \tag{9}$$

where $Q(\cdot)$ is the Q-function defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} \exp\Big(-\frac{u^2}{2}\Big)\,\mathrm{d}u. \tag{10}$$

Algorithm 1 Joint interleaver and irregular polar codes design

Initialize:
$\widetilde{\boldsymbol{\mathcal{C}}} = [\widetilde{C}_1, \widetilde{C}_2, ..., \widetilde{C}_N]$: mutual information of each modulated bit at eigen-mode channels for an ave. SNR of ρ
Start:
for all interleaver sets $\mathit{\Pi}$ in consideration **do**
  perform de-interleaving: $\boldsymbol{\mathcal{I}} = \mathit{\Pi}^{-1}(\widetilde{\boldsymbol{\mathcal{C}}})$
  activate all polarization units
  while $N_\mathrm{inact} \in \{1, 2, \ldots, N_\mathrm{U}\}$ **do**
    for all active polarization units **do**
      inactivate the target polarization unit
      $\boldsymbol{\mathcal{I}}' = \mathsf{UpdateMI}(\boldsymbol{\mathcal{I}})$ according to (7)
      select frozen bits $\bar{\mathbb{K}}$ having the $N-k$ smallest $\boldsymbol{\mathcal{I}}'$
      calculate the upper bound P_e according to (9)
      reactivate the target polarization unit
    end for
    inactivate the polarization unit having smallest P_e
  end while
end for
Return: best interleaver, frozen bit locations, and inactivated polarization units achieving the smallest P_e

Note that the EXIT evolution in (7) assumed SC decoding, where extrinsic information at the $(l-1)$th polarization stage are not propagated back to the lth polarization stage. Nonetheless, the EXIT evolution can be readily modified for belief-propagation decoding as well.

B. Joint Inactivation, Frozen Bit, and Interleaver Design

We describe our proposed design method of irregular polar codes for BICM and massive MIMO in Algorithm 1. Given an input mutual information array $\widetilde{\boldsymbol{\mathcal{C}}} = [\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_N]$ for every bit of modulated symbols s_i in eigen-mode MIMO channels at an average signal-to-noise ratio (SNR) ρ, the algorithm performs joint optimization of inactivation, frozen bit, and interleaver to minimize the union bound P_e. Since a brute-force search for all possible inactivations is unrealistic (i.e., the maximum search space is scaled to $2^{N_\mathrm{U}} = N^{N/2}$), we use a greedy method, which looks for the best polarization unit to be inactivated in a successive manner until we inactive N_inact polarization units which achieved the smallest P_e.

Since the polarization phenomenon depends highly on the mapping of the coded bits x_i to modulation bits in fading channels as discussed in [7–9], we analyze every one of the interleavers under consideration, and first de-interleave the mutual information array $\widetilde{\boldsymbol{\mathcal{C}}}$ as $\boldsymbol{\mathcal{I}} = \Pi^{-1}(\widetilde{\boldsymbol{\mathcal{C}}})$. For each interleaver Π, the mutual information $\boldsymbol{\mathcal{I}}$ are updated to obtain the output mutual information $\boldsymbol{\mathcal{I}}'$ according to evolution in (7) for each candidate of inactivation. The frozen bit locations $\bar{\mathbb{K}}$ are decided by choosing $N-k$ indices having the smallest values in $\boldsymbol{\mathcal{I}}'$. For each iteration up to N_{inact} inactivations, the next polarization unit to be inactivated is successively decided by analyzing the union bound P_{e} as in (9). Through the iterations over interleavers and successive inactivation, we finally select the best irregular polar code with the set of inactivated units and interleaver as well as frozen bit locations, that result in the minimum P_{e} as the output of Algorithm 1. Although the global optimum is not guaranteed, excellent performance was empirically observed with this greedy method.

C. Interleaver for Massive MIMO Channels

Appropriate design of interleavers in BICM schemes can result in considerable gain especially in wireless fading scenarios, as discussed in [7–9]. We consider two types of interleavers in Algorithm 1: block interleavers and quadratic polynomial permutation (QPP) interleavers [8, 9]. Both have been used in wireless standards for turbo coding due to their practical advantages. Among others these include a high degree of parallelism due to its maximum contention-free property, and high flexibility despite a relatively small number of parameters to be optimized. It is reported in [8, 9] that the QPP interleaver (originally introduced for turbo codes) with appropriate parameters can achieve significant performance gains for polar-coded BICM when compared to block interleavers over wireless fading channels.

When block interleavers are used, coded bits are first written row-by-row to a block of size $\mathcal{B} \times \frac{N}{\mathcal{B}}$ where N is divisible by $\mathcal{B}$. Next, the interleaved coded bits are retrieved column-by-column from the block. For interleaver design in Algorithm 1, we consider all possible block sizes $\mathcal{B} \leq N$ that are integer powers-of-two. When using QPP interleavers, the ith coded bit is interleaved as

$$\Pi(i) = \left\{ \left(f_0 + f_1(i-1) + f_2(i-1)^2\right) \bmod N \right\} + 1, \quad (11)$$

where f_0, f_1, and f_2 are the interleaver coefficients to be optimized under the constraints that f_1 is co-prime to N and f_2 contains all prime factors of N. In Algorithm 1, we consider all possible QPP interleavers over $0 \leq f_0 \leq 3$, $0 \leq f_1 \leq 71$ and $0 \leq f_2 \leq N$.

In order to design irregular polar codes for massive MIMO channels, we need to know the eigenvalue distribution of the Gram channel matrix $\boldsymbol{H}^\dagger \boldsymbol{H}$ to calculate the mutual information array $\widetilde{\boldsymbol{\mathcal{C}}}$ for every spatially multiplexed modulation. It is known from random matrix theory [26] that the probability density function (PDF) of eigenvalues asymptotically converges to $f(\nu) = \frac{\sqrt{2-\nu}}{2\pi\nu}$ for uncorrelated MIMO Rayleigh fading channels. Although this asymptotic PDF does not provide the exact PDF for the case with a finite number of antennas and correlated fading channels, it may still be useful for code construction in an MIMO system using a moderately massive number of antennas. The asymptotic cumulative distribution function (CDF) of the eigenvalues is expressed as follows:

$$F(\nu) = \frac{\sqrt{\nu(4-\nu)} + 4\sin^{-1}(\sqrt{\nu/4})}{2\pi}. \quad (12)$$

We use a precoding matrix $\boldsymbol{D}$ which allocates the modulated symbols to the instantaneous eigen-modes in an ascending order from the worst to best. Supposed that eigenvalues are independent, the instantaneous SNR ρ_i of the ith worst symbol (of the ith worst eigen-mode) follows an order statistics:

$$\Pr(\rho_i \leq \nu) = \sum_{j=i}^{N'} \binom{N'}{j} \big[F(\nu)\big]^j \big[1 - F(\nu)\big]^{N'-j}. \quad (13)$$

Based on this distribution, we can calculate the expectation of the bit-wise mutual information for every symbol to obtain $\widetilde{\boldsymbol{\mathcal{C}}}$.

D. Encoding and Decoding

Even with inactivation of polarization units in irregular polar codes, modification for encoding and decoding is not cumbersome as it requires no computation for inactivated polarization units. Therefore, the encoding and decoding complexity can be significantly reduced from the total number of polarization units N_{U} to the number of active units $N_{\mathrm{U}} - N_{\mathrm{inact}}$.

Although we use SC list decoding with CRC [10] in this paper, various reduced-complexity decoding methods [12–16] are available. Among them, our proposed polar codes with pruned polarization units are partly related to a tree-based list decoding [16], in which the decoding complexity is reduced by pruning/merging tree branches involving frozen bits to avoid unnecessary computation. However, our work directly modifies the degree distribution of polar generator matrix to drastically reduce both the encoding and decoding complexity without limitation of the tree structure regardless of frozen bits, and has the flexibility to incorporate any other methods including [16] to further reduce the decoding complexity.

It is known that non-binary kernels [27, 28] can yield faster rates of polarization than the conventional binary kernel, with the cost of increasing complexity. We note that the concept of irregular polar coding is applicable for non-binary kernels as well, by either pruning the kernel or replacing the kernel with other full-rank kernels in an irregular fashion. We also note that the method of systematic polar coding in [29] remains applicable to our irregular polar coding.

V. Performance Results

We evaluate the performance of the proposed irregular polar codes in this section. We first validate our EXIT-based design method in BICM to show the benefit achieved by exploiting non-uniform bit-reliability. We then analyze the computational complexity and BER performance of irregular polar codes in massive MIMO channels, to compare with relaxed polar codes [18] and conventional regular polar codes.

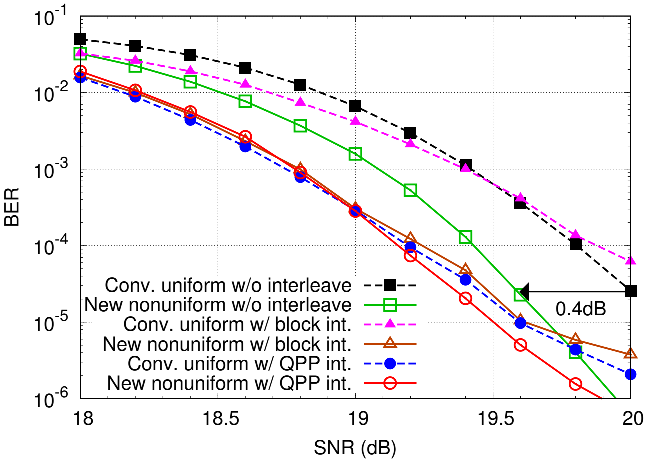


Fig. 4. BER of polar-coded 256QAM in AWGN channels.

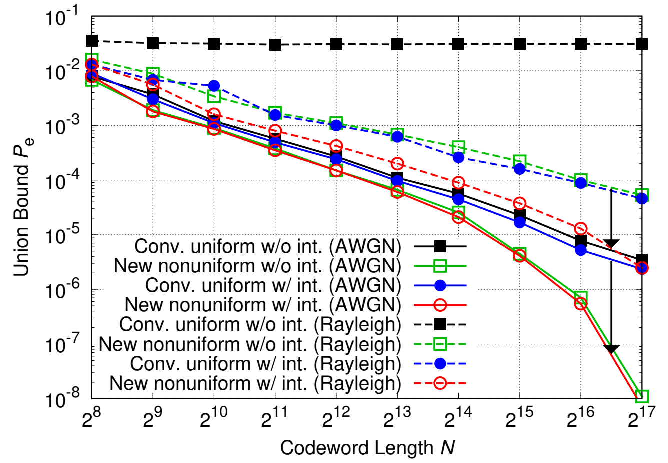


Fig. 6. Union bound P_{e} as a function of codeword length N for polar-coded 256QAM in AWGN channel at 19 dB and Rayleigh fading channel at 22 dB.

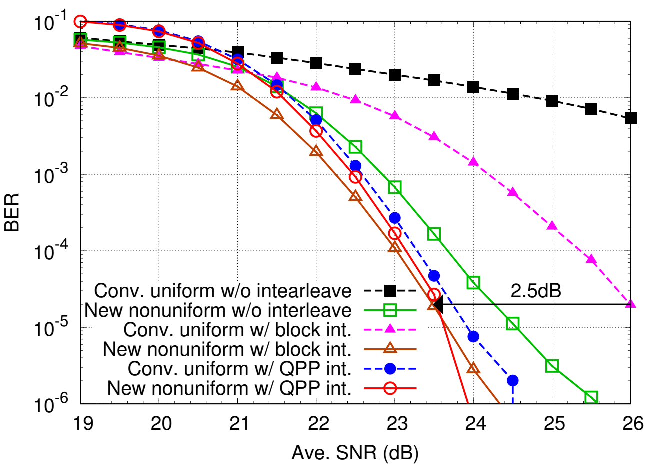


Fig. 5. BER of polar-coded 256QAM in Rayleigh fading channels.

A. Non-Uniformity-Aware Polar-Coded BICM Design

We demonstrate that a performance gain is attained by exploiting the non-uniformity of coded bits for BICM transmission in AWGN and Rayleigh fading channels. In Rayleigh fading channels, the symbol SNRs follow an exponential distribution whose CDF is $F(\nu) = 1 - \exp(-\nu/\rho)$. We compare the design methods with and without considering the non-uniformity of bit-LLRs. Note that the conventional construction method relies on an assumption of uniform bit reliability via idealistic interleaving to realize averaged mutual information for optimizing frozen bit locations.

We employ a systematic regular polar code with $N = 1024$ and code rate 0.8 with 8-bit CRC for 256QAM transmission. We use a list size of 32 for decoding. The BER performance for AWGN and Rayleigh fading channels are shown in Figs. 4 and 5, respectively, where the best QPP parameters were $(f_0, f_1, f_2) = (0, 7, 0)$ and $(2, 7, 16)$. It is demonstrated that significant gain can be realized when we take into consideration the non-uniform bit reliabilities. Comparing Fig. 5 with Fig. 4, it was verified that the performance gains of our method in fading channels are more significant than in AWGN channels because extreme non-uniformity in bit-LLRs is inherent in selective fading.

More importantly, we found that our new design method can improve the rate of polarization by taking the non-uniform reliability into consideration for code construction. This can be seen in Fig. 6, where we plot the union bounds P_{e} as a function of the codeword lengths N. In particular for fading channels, we observe a significant improvement in polarization rate from the joint optimization of frozen bit locations and interleaver.

B. Irregular Polar Codes for Massive MIMO

We explore the benefit of our design method for irregular polar codes in massive MIMO channels to evaluate the BER improvement and the complexity reduction in comparison to the conventional regular polar codes and relaxed polar codes [18]. We consider a $(256, 128)$ polar code with a rate of 0.5 for 64×64 MIMO transmission with 256QAM.

Fig. 7 presents the union bound P_{e} vs. the number of inactivated polarization units N_{inact} for the greedy design method. Note that $N_{\mathrm{inact}} = 0$ corresponds to conventional regular polar codes, whereas $N_{\mathrm{inact}} = N_{\mathrm{U}} = 1024$ corresponds to uncoded case. It is confirmed that our proposed irregular polar codes can drastically reduce computational complexity by up to 52% with a marginal improvement in the union bound, whereas relaxed polar codes achieve little complexity reduction since fewer degrees are available for selecting inactivations when the code construction method cannot handle non-uniformity.

Finally, we confirm that our irregular polar codes with reduced complexity have no penalty in performance in Fig. 8, where the BER performance with a list size of $L = \{1, 32\}$ is shown. Here, $N_{\mathrm{inact}} = 397$ units are inactivated in the best irregular polar code (with a complexity reduction by 38.8%), while 156 units are inactivated in the best relaxed polar codes (with a complexity reduction by 15.2%). It can be seen that the BER of the irregular polar codes is comparable to that of

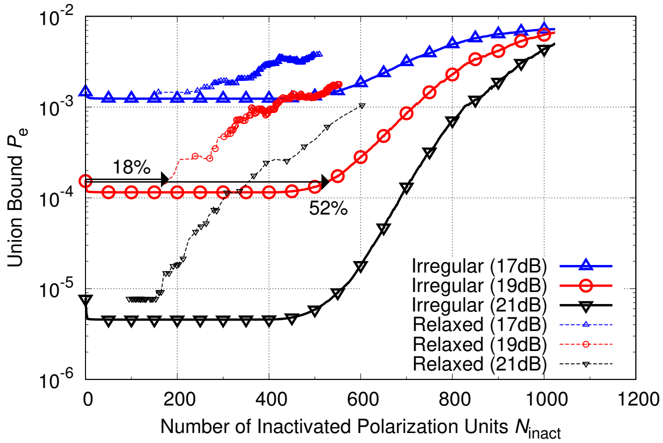


Fig. 7. Union bound $P_{\rm e}$ of regular, relaxed, and irregular polar codes for $N = 256$ and $k = 128$ in 64×64 MIMO 256QAM transmission.

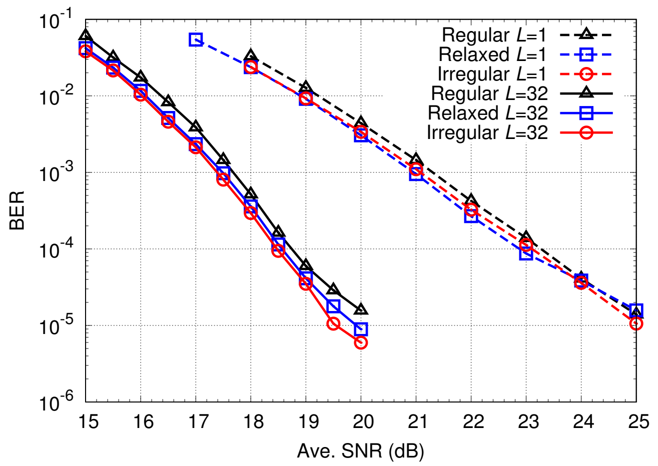


Fig. 8. BER performance of regular, relaxed, and irregular polar codes in 64×64 MIMO 256QAM transmission.

relaxed polar codes, and slightly better than the conventional polar codes while also reducing complexity.

VI. Conclusion

We proposed irregular polar codes having irregular inactivations of polarization units. With careful inactivation of polarization units, the computational complexity for encoding and decoding can be significantly reduced and meanwhile the BER performance can be improved because we can flatten the reliability to improve the worst non-frozen bit. We introduced an EXIT-based code construction to address non-uniform bit reliability to jointly optimize inactivation, frozen bit locations, and interleaver. The proposed design method showed an improved rate of polarization in particular for fading channels. We demonstrated that the irregular polar codes perform better than the conventional regular polar codes in massive MIMO transmission while significantly reducing complexity by 50%.

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