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**Abstract**

Stochastic Model Predictive Control (SMPC) accounts for model uncertainties and disturbances based on their statistical description. SMPC is synergistic with the well-established fields of stochastic modeling, stochastic optimization, and estimation. In particular, SMPC benefits from availability of already established stochastic models in many domains, existing stochastic optimization techniques, and wellestablished stochastic estimation techniques. For instance, the effect of wind gusts on an aircraft can be modeled by stochastic von Karman and Dryden's models but no similar deterministic models appear to exist. Loads or failures in electrical power grids, prices of financial assets, weather (temperature, humidity, wind speed and directions), computational loads in data centers, demand for a product in marketing/supply chain management are frequently modeled stochastically thereby facilitating the application of the SMPC framework. A comprehensive overview of various approaches and applications of SMPC has been given in the article. Another overview article in Encyclopedia of Systems and Control is focused on tube SMPC approaches. This chapter provides a tutorial exposition of several SMPC approaches.

*Handbook of Model Predictive Control*

# Stochastic Model Predictive Control

Ali Mesbah, Ilya Kolmanovsky and Stefano Di Cairano

## I. Introduction

Stochastic Model Predictive Control (SMPC) accounts for model uncertainties and disturbances based on their statistical description. SMPC is synergistic with the well-established fields of stochastic modeling, stochastic optimization, and estimation. In particular, SMPC benefits from availability of already established stochastic models in many domains, existing stochastic optimization techniques, and well-established stochastic estimation techniques. For instance, the effect of wind gusts on an aircraft can be modeled by stochastic von Kármán and Dryden's models [21] but no similar deterministic models appear to exist. Loads or failures in electrical power grids, prices of financial assets, weather (temperature, humidity, wind speed and directions), computational loads in data centers, demand for a product in marketing/supply chain management are frequently modeled stochastically thereby facilitating the application of the SMPC framework.

A comprehensive overview of various approaches and applications of SMPC has been given in the article [33]. Another overview article [27] in *Encyclopedia of Systems and Control* is focused on tube SMPC approaches. This chapter provides a tutorial exposition of several SMPC approaches.

## II. Stochastic Optimal Control and MPC with Chance Constraints

Consider a stochastic, discrete-time system,

$$x_{t+1} = f(x_t, u_t, w_t), \tag{1a}$$

$$y_t = h(x_t, v_t), \tag{1b}$$

where $t$ is the time index; $x_t \in \mathbb{R}^{n_x}$, $u_t \in \mathbb{R}^{n_u}$, and $y_t \in \mathbb{R}^{n_y}$ are the system states, inputs, and outputs, respectively; $w_t \in \mathbb{R}^{n_w}$ denotes stochastic system noise; $v_k \in \mathbb{R}^{n_v}$ denotes measurement noise; and functions

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$f:\mathbb{R}^{n_x}\times\mathbb{R}^{n_u}\times\mathbb{R}^{n_w}\to\mathbb{R}^{n_x}$ and $h:\mathbb{R}^{n_x}\times\mathbb{R}^{n_v}\to\mathbb{R}^{n_y}$ define system state and output equations, respectively. The uncertain initial state $x_0$ is described by the known probability distribution $\mathsf{P}[x_0]$. The independent and identically distributed random variables in the noise sequences $\{w_t\}$ and $\{v_t\}$ have known probability distributions $\mathsf{P}[w]$ and $\mathsf{P}[v]$, respectively.

The system (1) represents a Markov decision process, as the successor state $x_{t+1}$ can be determined from the current state $x_t$ and input $u_t$ [29]. Let $I_t$ denote the vector of system information that is causally available at time instance $t$,

$$I_t := [y_t,\ldots,y_0,u_{t-1},\ldots,u_0],$$

with $I_0 := [y_0]$. The conditional probability of state $x_t$ given $I_t$, i.e., $\mathsf{P}[x_t|I_t]$, can be computed via *recursive Bayesian estimation* [15]

$$\mathsf{P}[x_t|I_t] = \frac{\mathsf{P}[y_t|x_t]\mathsf{P}[x_t|I_{t-1}]}{\mathsf{P}[y_t|I_{t-1}]}, \tag{2a}$$

$$\mathsf{P}[x_{t+1}|I_t] = \int \mathsf{P}[x_{t+1}|x_t,u_t]\mathsf{P}[x_t|I_t]dx_t, \tag{2b}$$

with $\mathsf{P}[x_0|I_{-1}] := \mathsf{P}[x_0]$. We use $\mathbb{E}_{x_t}$ and $\mathbb{P}_{x_t}$ to denote, respectively, the expected value and probability with respect to the stochastic state $x_t$ (with uncertainty $\mathsf{P}[x_t|I_t]$) as well as the random variables $w_k$ and $v_k$ for all $k>t$.

Let $N\in\mathbb{N}$ be the prediction horizon.[1] Consider an $N$-stage control policy

$$\Pi := \{\pi_0,\pi_1,\ldots,\pi_{N-1}\}, \tag{3}$$

where $\pi_k\in\mathbb{U}\subset\mathbb{R}^{n_u}$ is a nonanticipatory feedback control law; and $\mathbb{U}$ is a nonempty measurable set for the inputs. At the $k$th stage of control, $u_k=\pi_k$. Define the control cost function as

$$J_N(x_t,\Pi) = \mathbb{E}_{x_t}\big[c_{t+N}(x_{t+N}) + \sum_{k=0}^{N-1} c_{t+k}(x_{t+k},\pi_{t+k})\big], \tag{4}$$

where $c_{t+k}:\mathbb{R}^{n_x}\times\mathbb{U}\to[0,\infty)$ and $c_{t+N}:\mathbb{R}^{n_x}\to[0,\infty)$ denote the stage-wise cost incurred at the $(t+k)$th stage of control and at the terminal stage, respectively. Define a *joint chance constraint* of the form

$$\mathbb{P}_{x_{t+k}}[g(x_{t+k})\leq 0] \geq 1-\delta, \quad k=1,\ldots,N, \tag{5}$$

where $g:\mathbb{R}^{n_x}\to\mathbb{R}^{n_c}$ denotes state constraints, composed of $n_c>1$ inequalities; and $\delta\in[0,1)$ denotes the maximum allowed probability of state constraint violation. The chance constraint (5) is generally nonconvex and intractable [3], [7]; see [36], [18] for treatment of chance-constrained optimization.

[1]For notational convenience, the control and prediction horizons are considered to be identical.

*Remark 1:* When the probabilistic uncertainties in (1) are bounded, hard state constraints can be enforced by setting $\delta = 0$ in (5). This implies that $g(x_t) \leq 0$ must be satisfied for all realizations of system uncertainties.

Given the (uncertain) knowledge of the system state at sampling time $t$, i.e., $\mathsf{P}[x_t|I_t]$, the stochastic optimal control problem (OCP) for system (1) is stated as

$$\min_{\Pi} \; J_N(x_t, \Pi) \tag{6a}$$

$$\text{s.t.:} \; x_{t+k+1|t} = f(x_{t+k|t}, \pi_{t+k|t}, w_{k|t}), \qquad k = 0, \ldots, N-1, \tag{6b}$$

$$\pi_{t+k|t} \in \mathbb{U}, \qquad k = 0, \ldots, N-1, \tag{6c}$$

$$\mathbb{P}_{x_{t+k|t}}[g(x_{t+k|t}) \leq 0] \geq 1 - \delta, \qquad k = 1, \ldots, N, \tag{6d}$$

$$w_{k|t} \sim \mathsf{P}[w], \qquad k = 0, \ldots, N-1, \tag{6e}$$

$$x_{t|t} \sim \mathsf{P}[x_t|I_t], \tag{6f}$$

where $t+k|t$ denotes the state and input computed at time $t+k$ based on the knowledge of $\mathsf{P}[x_t|I_t]$. Note that the chance constraint (6d) enforces the state constraints with respect to $\mathsf{P}[x_t|I_t]$ as well as the future noise sequence over the horizon $N$.

In theory, the stochastic OCP (6) can be solved offline using *Bellman's principle of optimality* [2]. The resulting optimal control policy $\Pi^*$ can then be implemented in a receding-horizon manner by applying $u_t = \pi_t^*$ to the stochastic system (1) at every sampling time $t$ that $\mathsf{P}[x_t|I_t]$ is estimated from (2). The principle of optimality requires that the optimal control cost at each control stage satisfy the Bellman equation for stochastic dynamic programming. To this end, the control input $\pi_t$ at each stage $t$ must be designed via a nested minimization of the expected sum of the current control cost and the optimal future control cost, which is computed based on the knowledge of the future state $\mathsf{P}[x_{t+1}|I_{t+1}]$ (e.g., see [34]). Although solving the Bellman equation will result in an optimal *closed-loop* control policy, it is well-known that stochastic dynamic programming suffers from the so-called *curse of dimensionality* for practically-sized systems [6].

In recent years, a plethora of SMPC strategies have been presented that seek online solution of an approximate surrogate for the stochastic OCP (6) in a receding-horizon manner. Generally speaking, SMPC strategies neglect the effect of the control input $\pi_t$ on the knowledge of the future state $\mathsf{P}[x_{t+1}|I_{t+1}]$ to avoid the formidable challenge of solving the Bellman equation [26], [33]. In the remainder of this chapter, three SMPC strategies are introduced for receding-horizon control of stochastic linear systems.

## III. Scenario tree-based MPC

The scenario tree-based stochastic MPC approach was introduced in [4], and relies upon multi-stage stochastic programming [9]. The general idea behind tree-based stochastic MPC is to compute a closed-loop policy based on scenarios determined by predictions of the stochastic disturbance sequences. Due to causality, the predicted states and related control sequences result arranged in a tree structure. A first application of this methodology, in the context of robust min-max MPC was in [45]. In the stochastic context, each tree node is further associated with a probability of reaching it, based on the probability of the scenario to realize. Such a probability can be used to selectively trim parts of the tree that are unlikely to realize in order to reduce the computational effort. For other scenario-based approaches, see [10], [44] and references in [33].

In this section we modify the notations slightly, reserving subscripts to designate the nodes of the scenario tree, and using $x(t)$, $u(t)$, $w(t)$, etc. to denote the variables at the current time instant, $t$.

The system is modeled as a parameter varying discrete-time linear system, possibly with an additive disturbance,

$$x(t+1) = A(w(t))x(t) + B(w(t))u(t) + F(w(t)), \tag{7}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the input, and $w(t) \in \mathscr{W}$ is a scalar stochastic disturbance, which takes values in a finite set $\{\bar{w}_1, \dots, \bar{w}_s\} \subset \mathbb{R}$. The state and input vectors in (7) are subject to the pointwise-in-time constraints

$$x(t) \in \mathscr{X},\ u(t) \in \mathscr{U},\ \forall t \in \mathbb{Z}_{0+}, \tag{8}$$

which must hold for all $t \geq 0$, where $\mathscr{X} \subseteq \mathbb{R}^{n_x}$, $\mathscr{U} \subseteq \mathbb{R}^{n_u}$, are polyhedral sets. The probability mass function $p(t)$ of $w$ is assumed to be known or predictable, at all times, that is, for all $t \in \mathbb{R}_{0+}$, $p_j(t) = \Pr[w_j(k) = \bar{w}_j]$, such that $p_j(t) \geq 0$, $\sum_{j=1}^{s} p_j(t) = 1$ is known, and it can be predicted for $\tau > t$ based only on the information known at time $t$. This includes the cases when $p(t)$ is constant, or varies in a pre-defined way, or when it is defined by a stochastic Markov process with state $z$, and $z(t+1) = f_M(z(t))$, $p(t) = p(z(t))$, where $z(t)$ is known at time $t$. In the latter case, the disturbance realization in $\mathscr{W}$ represents the combinations of disturbances on the system and the (discrete) transitions of the Markov process. The main restriction imposed by this assumption is that $p$ cannot depend on the system state $x$, since the system evolution is affected by the input $u$, and hence $p(\tau)$, $\tau > t$, will not be predictable based only on data at time $t$.

### A. Scenario-tree Construction

Due to the presence of the stochastic disturbance, the MPC aims at minimizing the expected value of a given cost function, that is

$$\mathbb{E}[J_N(t)] = \mathbb{E}\left[\sum_{k=1}^{N} x(t+k|t)^\top Q x(t+k|t) + \sum_{k=0}^{N-1} u(t+k)^\top R u(t+k)\right], \tag{9}$$

where $N \in \mathbb{Z}_+$ is the prediction horizon, and $Q = Q^\top \geq 0$, $R = R^\top > 0$ are weight matrices of appropriate dimensions. Since $|\mathscr{W}|$ is finite, $p(t)$ is known, and $p(t+k)$ can be predicted based only on the information at time $t$, one can enumerate all the admissible realizations of the stochastic disturbance sequence along the finite horizon $N$, and their corresponding probabilities. An $N$-steps scenario, $\omega_\ell^N \in \mathscr{W}^N$, is a sequence of $N$ disturbance realizations $\omega^N = [w(0), \ldots, w(N-1)]$, and its $q$ steps prefix $\omega^{N,q}$ is the subsequence composed of only its first $q$ elements $\omega^{N,q} = [w(0), \ldots, w(q-1)]$. Thus, one can optimize (9) by optimizing

$$\mathbb{E}[J_N(t)] = \sum_{\ell=1}^{s^N} J_N(t|\omega_\ell^N(t)) \mathsf{P}(\omega_\ell^N(t)|z(t)),$$

with constraints that enforce causality, i.e., $u(t+k|\omega_j^N) = u(t+k|\omega_i^N)$ for all $i, j$ such that $\omega_i^{N,k} = \omega_j^{N,k}$. However, the optimization problem obtained in this way is large, because it considers all disturbance sequences, even those that occur with arbitrarily small probability.

In the scenario tree-based MPC, (7), (8), and the predicted evolution of $p(t+k)$ are used to construct a variable horizon optimization problem where only the disturbance sequences that are more likely to realize are accounted for, and hence the size of the optimization problem is reduced. The scenario tree describes the most likely scenarios of future disturbance realizations, and is updated at every time step using newly available measurements of the state $x(t)$, and updated information to predict the disturbance probability $p(t+k)$. In order to explain the scenario tree-based approach, we introduce the following notations:

- $\mathscr{T} = \{\mathscr{N}_1, \mathscr{N}_2, \ldots, \mathscr{N}_n\}$: the set of the tree nodes. Nodes are indexed progressively as they are added to the tree, i.e., $\mathscr{N}_1$ is the root node and $\mathscr{N}_n$ is the last node added;
- $pre(\mathscr{N}) \in \mathscr{T}$: the predecessor of node $\mathscr{N}$;
- $succ(\mathscr{N}, w) \in \mathscr{T}$: the successor of node $\mathscr{N}$ for $w \in \mathscr{W}$;
- $\pi_{\mathscr{N}} \in [0,1]$: the probability of reaching $\mathscr{N}$ from $\mathscr{N}_1$;
- $x_{\mathscr{N}} \in \mathbb{R}^{n_x}$, $u_{\mathscr{N}} \in \mathbb{R}^{n_u}$, $w_{\mathscr{N}} \in \mathscr{W}$: the state, input, and disturbance value, respectively, associated with node $\mathscr{N}$, where $x_{\mathscr{N}_1} = x(t)$, and $w_{\mathscr{N}_1} = w(t)$;
- $\mathscr{C} = \{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_c\}$: the set of candidate nodes, i.e., $\mathscr{C} = \{\mathscr{N} \notin \mathscr{T} \mid \exists (i,j) : \mathscr{N} = succ(\mathscr{N}_i, w_j)\}$;

- $\mathscr{S} \subset \mathscr{T}$: the set of leaf nodes, with cardinality denoted by $n_{\text{leaf}} = |\mathscr{S}|$, i.e., $\mathscr{S} = \{\mathscr{N} \in \mathscr{T} \mid succ(\mathscr{N}, w_j) \notin \mathscr{T},\ \forall j \in \{1, \ldots, s\}\}$.

Every path from the root node to a leaf node is a scenario in the tree and describes a disturbance realization that will be accounted for in the optimization problem. The procedure to construct the scenario tree is as follows.

Starting from the root node $\mathscr{N}_1$, which is associated with $w(t)$, we construct a list $\mathscr{C}$ of candidate nodes by considering all the possible $s$ future values of the disturbance in $\mathscr{W}$ and their realization probabilities. The candidate with maximum probability $\mathscr{C}_{i^*}$ is added to the tree and removed from $\mathscr{C}$. The procedure is repeated by adding at every step new candidates as children of the last node added to the tree, until the tree contains $n_{\max}$ nodes. The scenario-tree construction, summarized in Algorithm 1, expands the tree in the most likely direction, so that the paths with higher probability are extended longer in the future, because they may have larger impact on the overall performance. This leads to a tree with variable depth, where the paths from the root to the leaves may have different lengths and hence different prediction horizons, see Fig. 1.

**Algorithm 1** SMPC tree generation procedure

At any step $k$:<br>
set $\mathscr{T} = \{\mathscr{N}_1\}$, $\pi_{\mathscr{N}_1} = 1$, $n = 1$, $c = s$;<br>
set $\mathscr{C} = \bigcup_{j=1}^{s} \{succ(\mathscr{N}_1, w_j)\}$<br>
**while** $n < n_{\max}$ **do**<br>
    **for all** $i \in \{1, 2, \ldots, c\}$, **do**<br>
        compute $\pi_{\mathscr{C}_i}$ ;<br>
    **end for**<br>
    set $i^* = \arg\max_{i \in \{1,2,\ldots,c\}} \pi_{\mathscr{C}_i}$;<br>
    set $\mathscr{N}_{n+1} = \mathscr{C}_{i^*}$;<br>
    set $\mathscr{T} = \mathscr{T} \cup \{\mathscr{N}_{n+1}\}$;<br>
    set $\mathscr{C} = \bigcup_{j=1}^{s} \{succ(\mathscr{C}_{i^*}, w_j)\} \cup (\mathscr{C} \setminus \mathscr{C}_{i^*})$;<br>
    set $c = c + s - 1$, $n = n + 1$;<br>
**end while**

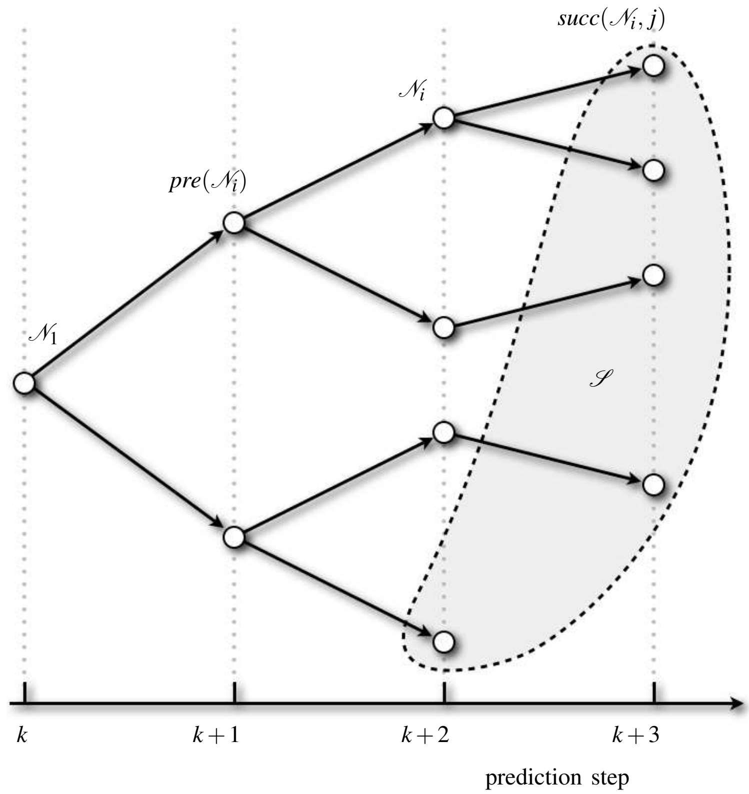


Fig. 1. Graphical representation of a multiple-horizon optimization tree. Some roof-to-leaves paths have length 2, others have length 3. Hence, different scenarios may have different prediction horizons.

### *B. Scenario-tree Stochastic Optimization Problem*

The scenario-tree constructed with Algorithm 1 is exploited to repeatedly construct the MPC optimization problem. For the sake of notation, in what follows we use $x_i$, $u_i$, $y_i$, $w_i$, $\pi_i$ and $pre(i)$ to denote $x_{\mathscr{N}_i}$, $u_{\mathscr{N}_i}$, $y_{\mathscr{N}_i}$, $w_{\mathscr{N}_i}$, $\pi_{\mathscr{N}_i}$ and $pre(\mathscr{N}_i)$, respectively.

Given the maximum number of nodes, $n_{\max}$, at any time $t$, the Scenario-tree based Stochastic MPC performs the following operations: $(i)$ it constructs the tree $\mathscr{T}(t, n_{\max})$ based on $w(t)$; $(ii)$ it solves the

the following stochastic MPC based on $\mathscr{T}(t,n_{\max})$:

$$\min_{\mathbf{u}} \sum_{i\in\mathscr{T}(t,n_{\max})\setminus\{\mathscr{N}_1\}} \pi_i x_i^\top Q x_i + \sum_{i\in\mathscr{T}(t,n_{\max})\setminus\mathscr{S}} \pi_i u_i^\top R u_i \tag{10a}$$

$$\text{s.t. } x_1 = x(t) \tag{10b}$$

$$x_i = A_{pre(i)} x_{pre(i)} + B_{pre(i)} u_{pre(i)} + F_{pre(i)},\ i \in \mathscr{T}(t,n_{\max})\setminus\{\mathscr{N}_1\} \tag{10c}$$

$$x_i \in \mathscr{X},\ i \in \mathscr{T}(t,n_{\max})\setminus\{\mathscr{N}_1\} \tag{10d}$$

$$u_i \in \mathscr{U},\ i \in \mathscr{T}(t,n_{\max})\setminus\mathscr{S} \tag{10e}$$

where $\mathbf{u} = \{u_i : \mathscr{N}_i \in \mathscr{T}(t,n_{\max})\setminus\mathscr{S}\}$ is the multiple-horizon input sequence; $(iii)$ it applies $u(t) = u_1 = u_{\mathscr{N}_1}$ as a control input to the system (7).

In Problem (10), causality is enforced by the tree structure, since if $\omega_i^{N,k} = \omega_j^{N,k}$, the two scenarios share the same path in the tree, at least until the $k^{th}$ level, and since a single control input is associated to each node in the tree, it follows automatically that $u(t+h|\omega_j^N) = u(t+h|\omega_i^N)$ for all $h = 0,\ldots,k$. Indeed, causality is enforced based on equal disturbance sequences, as opposed to, for instance, [45], where causality is enforced based on reaching the same state, possibly by different disturbance sequences. Thus, the approach of (10) may result in redundant decision variables, but it allows for simpler formulation as an optimization problem.

In fact, Problem (10) is a quadratic program (QP) with $n_u(n_{\max} - n_{\text{leaf}})$ variables. If the scenario tree $\mathscr{T}$ is fully expanded, i.e., all the leaf nodes are at depth $N$ and all parent nodes have $s$ successors, which obviously requires $n_{\max} = 2^{N+1} - 1$, the objective function (10a) is equivalent to (9). Otherwise, (10a) is an approximation of (9) based on the scenarios with highest probability, and thus $n_{\max}$ determines the representativeness-complexity tradeoff of the approximation.

Based on Problem (10), the scenario-tree stochastic MPC results in a closed-loop prediction policy. Since there are multiple predictions, i.e, multiple tree nodes, of the predicted state values and to each a possibly different control action is associated, the control action at any predicted step changes as function of the disturbance realizations up to such step. Thus, the control action implicitly encodes feedback from the past disturbances.

### *C. Extensions and Applications*

The Scenario-tree based MPC is a fairly general framework that allows to solve stochastic MPC problems with a precision that related to the amount of available computational resources. Several extensions have been presented in the literature.

In terms of modeling, in [5] it is shown that $p(t)$ can be generated by several classes of stochastic processes, possibly in an approximate discretization, such as the generalized autoregressive conditional heteroskedasticity (GARCH), and Markov chains with transition matrix $T$, and emission matrix $E$, where $z$ is the Markov chain state and

$$T_{ij} = \mathsf{P}[z(t+1) = z_j | z(t) = z_i], \tag{11a}$$

$$E_{ij} = \mathsf{P}[w(t) = w_j | z(t) = z_i] = p_j(z_i). \tag{11b}$$

A simplified formulation of the Markov chain is the case where $z = w$, resulting in $\mathsf{P}[w(t+1)] = f_M(w(t))$so that $T_{ij} = P[w(t+1) = w_j | w(t) = w_i]$ and $E = I$.

For the case where $F(w) = 0$, for all $w \in \mathscr{W}$, and there are no (hard) constraints, uniform mean square exponential stability of the closed-loop system is demonstrated in [5] by designing offline a stochastic Lyapunov function satisfying $\mathscr{V}(x) = x^{\mathrm{T}} S x$, $\mathbb{E}[V(x(t+1))] - V(x(t)) \leq x(t)^{\mathrm{T}} L x(t)$, where $S, L > 0$, which is then enforced as constraint at the root node of the scenario tree $\mathscr{N}_1$ in Problem (10) by the quadratic constraint

$$\sum_{i=1}^{s} \mathsf{P}[w_i(k)](A_i x_1 + B_i u_1)^\top S (A_i x_1 + B_i u_1) \leq x_1^\top (S - L) x_1.$$

For the constrained case, in [5] it is suggested to construct an invariant ellipsoid [25]

$$\mathscr{E} = \{x : \; x^\top S x \leq \gamma\} \subset \mathscr{X},$$

and a linear controller $u = Kx$, such that $\mathscr{E}$ is robust positive positive invariant for the polytopic difference inclusion with vertices $[A(w), B(w)]$, $w \in \mathscr{W}$ controlled by $u = Kx$, and for all $x \in \mathscr{E}$, $Kx \in \mathscr{U}$. The invariant ellipsoid is exploited to construct another constraint to be added in Problem (10),

$$(A_i x_1 + B_i u_1)^\top S (A_i x_1 + B_i u_1) \leq \gamma, \; \forall i : \; p_i(t) > 0,$$

which guarantees recursive constraint satisfaction.

The scenario tree-based MPC is applied to energy management of a hybrid electric powertrain in [43]. In [16], once again motivated by the application to the energy management of hybrid electric powertains, the case where the stochastic disturbance is learned during execution has been presented. In particular, in [16] the actions of the vehicle driver are modeled as a Markov chain whose transition probability is time-varying and initially unknown, which is then estimated from the transition frequencies

$$\mathsf{P}[w(t+1) = w_j | w(t) = w_i] = \frac{n_{ij}}{n_i}, \tag{12}$$

with an iterative algorithm. The so-estimated Markov chain is used to adapt the scenario tree construction in the stochastic MPC for optimizing the energy efficiency of the hybrid electric vehicle, and it is shown that with the learning of the Markov chain, the overall performance is very close to the one from an MPC with exact preview, on both synthetic and experimental data. Along these lines, [8] reports an application of the scenario tree-based stochastic MPC to adaptive cruise control where the Markov chain is used to model the actions of the surrounding traffic,

In terms of numerical algorithms, the scenario tree-based MPC results in QPs that are larger than those from nominal MPC, but these have a special structure that can be exploited to reduce the computational burden. For instance, in [23] an algorithm based on the alternating direction method of multipliers (ADMM) was proposed, that exploits the structure and scales more favorably with the number of nodes in the tree than structure-ignoring algorithms, and allows for parallel implementation.

## IV. Polynomial chaos-based MPC

Polynomial chaos-based MPC strategies have been developed for receding-horizon control of stochastic linear [24], [39] and nonlinear systems [17], [1], [38] subject to probabilistic model uncertainty in initial conditions and parameters. The term *polynomial chaos* was introduced by Norbert Wiener in the seminal paper [47], in which a generalized harmonic analysis was applied to Brownian motion-like processes. The basic notion of polynomial chaos is to expand finite-variance random variables by an infinite series of Hermite polynomials, which are functions of a normally distributed input random variable [11]. The polynomial chaos framework has recently been generalized to non-Gaussian random variables by establishing the convergence properties of polynomials that are orthogonal with respect to possibly non-Gaussian input random variables [48]. The orthogonality of polynomials in generalized polynomial chaos (gPC) enables obtaining sample-free, closed-form expressions for propagation of high-order moments of states through the system dynamics. Alternatively, polynomial chaos expansions can be used as a surrogate for the system model for performing Monte Carlo simulations efficiently via algebraic operations in order to construct the probability distribution of states. This section uses gPC to present a sample-free formulation for SMPC of stochastic linear systems with probabilistic model uncertainty.

### *A. System model, constraints, and control input parameterization*

Consider a stochastic, linear system described by the prediction model

$$x_{t+k+1|t} = A(\theta)x_{t+k|t} + B(\theta)u_{t+k|t} + Dw_{t+k|t}, \tag{13}$$

where $\theta \in \mathbb{R}^{n_\theta}$ denotes the unknown system parameters that are modeled as (time-invariant) probabilistic uncertainties with probability distribution $\mathsf{P}[\theta]$; and the stochastic noise $w_{t+k|t}$ is a zero-mean Gaussian process with covariance $\Sigma_w$. The notation in (13) is as in (1). The probability distribution of parameters, $\mathsf{P}[\theta]$, quantifies our subjective belief in the unknown parameters, whereas the parameters are fixed in the true system.

A joint chance constraint of the form (5) is imposed, where the state constraint $g(x_{t+k|t}) \leq 0$ takes the form of a polytope

$$\mathbb{P}_{x_{t+k|t}}[\ C^\top x_{t+k|t} \leq d\ ] \geq 1-\delta, \quad k=1,\ldots,N, \tag{14}$$

with $C \in \mathbb{R}^{n_x \times n_c}$ and $d \in \mathbb{R}^{n_c}$. The control cost function is defined as

$$J_N(x_t, \mathbf{U}) = \mathbb{E}_{x_t}\Big[\ \sum_{k=0}^{N-1} ||x_{t+k|t}||_Q^2 + ||u_{t+k|t}||_R^2\ \Big], \tag{15}$$

where $Q$ and $R$ are symmetric and positive definite weight matrices; and $\mathbf{U} := [u_{t|t}, \ldots, u_{t+N-1|t}]$ denotes the vector of control inputs over the prediction horizon. We choose to parameterize the control inputs $u_{t+k|t}$ as an affine function of state [20]

$$u_{t+k|t} = L_k x_{t+k|t} + m_k, \quad k=1,\ldots,N, \tag{16}$$

where $L_k \in \mathbb{R}^{n_u \times n_x}$ and $m_k \in \mathbb{R}^{n_u}$ denote the feedback gains and control actions over the prediction horizon, respectively. The affine-state feedback parameterization (16) allows to account for the effect of state feedback over the prediction horizon. The underlying notion of (16) is that the system state will be known at the future time instants. Thus, the controller will have the state/disturbance information when designing the future control inputs over the prediction horizon.

*Remark 2:* It is generally impossible to guarantee satisfaction of the input bounds, i.e., $u_t(x_t) \in \mathbb{U}$, when the stochastic noise $w_t$ is unbounded. To alleviate this shortcoming of (16), a *saturation function* can be incorporated into the affine feedback control policy to enable direct handling of hard input bounds in the presence of unbounded stochastic noise [22].

The key challenges in the above discussed setup for SMPC arise from: (i) propagation of the probabilistic model uncertainty and stochastic noise through the prediction model (13), and (ii) computational intractability of the joint chance constraint (14). A gPC-based uncertainty propagation method, however, can be used to obtain closed-form expressions for the mean and covariance of the predicted state as explicit functions of the control input. A moment-based surrogate is then presented for (14) in terms of the Mahalanobis distance [30], which is exact when the system state has a multivariate normal distribution.

## B. Generalized polynomial chaos for uncertainty propagation

GPC seeks to approximate a stochastic variable $\psi(\xi)$ in terms of a finite expansion of orthogonal polynomial basis functions

$$\psi(\xi) \approx \hat{\psi}(\xi) := \sum_{i=0}^{p} a_i \phi_i(\xi) = \mathbf{a}^\top \Lambda(\xi), \tag{17}$$

where $\mathbf{a} := [a_0, \ldots, a_p]^\top$ denotes the vector of expansion coefficients; $\Lambda(\xi) := [\phi_0(\xi), \ldots, \phi_p(\xi)]^\top$ denotes the vector of multivariate polynomial basis functions $\phi_i$ of maximum degree $l$ with respect to the random variables $\xi \in \mathbb{R}^{n_\xi}$; and $p+1 = \frac{(n_\xi+l)!}{n_\xi!l!}$ denotes the total number of expansion terms. The basis functions belong to the Askey scheme of polynomials, which includes a set of orthogonal basis functions in the Hilbert space defined on the support of the random variables. Thus, the basis functions $\phi_i$ must be chosen in accordance with the probability distribution of the random variables $\xi$, as established in [48]. The orthogonality of the basis functions implies that $\langle \phi_i(\xi), \phi_j(\xi) \rangle = \langle \phi_i^2(\xi) \rangle \delta_{ij}$, where $\langle h(\xi), g(\xi) \rangle = \int_\Omega h(\xi) g(\xi) \mathsf{P}[\xi] d\xi$ denotes the inner product induced by $\mathsf{P}[\xi]$ and $\delta_{ij}$ denotes the Kronecker delta function. Hence, the expansion coefficients $a_i$ in (17) can be obtained via

$$a_i = \frac{\langle \hat{\psi}(\xi), \phi_i(\xi) \rangle}{\langle \phi_i(\xi), \phi_i(\xi) \rangle},$$

which can be computed analytically for linear and polynomial systems [19].

For a particular realization of the stochastic system noise $w$ in (13), the polynomial chaos expansions (PCEs) (17) can be used for efficient propagation of the model uncertainty $\theta$ through (13). Propagation of model uncertainty will yield the probability distribution of state conditioned on the noise realization, i.e., $\mathsf{P}[\hat{x}_{t+k|t}|w]$, which can then be integrated over all possible realizations of $w$ to obtain the complete probability distribution of the (polynomial chaos-approximated) state

$$\mathsf{P}[\hat{x}_{t+k|t}] = \int_{-\infty}^{\infty} \mathsf{P}[\hat{x}_{t+k|t}|w] \mathsf{P}[w] dw. \tag{18}$$

When the distribution of stochastic noise, $\mathsf{P}[w]$, is Gaussian, the moments of the probability distribution $\mathsf{P}[\hat{x}_{t+k|t}]$ in (18) can be readily defined in terms of the coefficients of $\hat{x}_{t+k|t}$. To this end, we approximate each predicted state and control input as well as the unknown parameters in the system matrices $A(\theta)$ and $B(\theta)$ in (13) by PCEs of the form (17). Define $\tilde{x}_{i,t+k|t} = [a_{i_0,t+k|t}, \ldots, a_{i_p,t+k|t}]^\top$ and $\tilde{u}_{i,t+k|t} = [b_{i_0,t+k|t}, \ldots, b_{i_p,t+k|t}]^\top$ to denote the coefficients of PCEs for the $ith$ predicted state and control input, respectively. The coefficients of PCEs for all states and control inputs can be concatenated into vectors $\tilde{\mathbf{x}}_{t+k|t} := [\tilde{x}_{1,t+k|t}^\top, \ldots, \tilde{x}_{n_x,t+k|t}^\top]^\top \in \mathbb{R}^{n_x(p+1)}$ and $\tilde{\mathbf{u}}_{t+k|t} := [\tilde{u}_{1,t+k|t}^\top, \ldots, \tilde{u}_{n_u,t+k|t}^\top]^\top \in \mathbb{R}^{n_u(p+1)}$, respectively. Using the Galerkin projection [19], the error in a gPC-based approximation of the prediction model (13), which arises from truncation in PCEs, can be projected onto the space of the multivariate basis functions $\{\phi_i\}_{i=0}^{p}$. This allows for expressing the prediction model (13) in terms of the coefficients of the PCEs for states and control inputs

$$\tilde{\mathbf{x}}_{t+k+1|t} = \mathbf{A}\tilde{\mathbf{x}}_{t+k|t} + \mathbf{B}\tilde{\mathbf{u}}_{t+k|t} + \mathbf{D}w_{t+k|t}, \tag{19}$$

where

$$\mathbf{A} = \sum_{i=0}^{p} A_i \otimes \Psi_i; \quad \mathbf{B} = \sum_{i=0}^{p} B_i \otimes \Psi_i; \quad \mathbf{D} = D \otimes e_{p+1};$$

$$\Psi_i := \begin{bmatrix} \sigma_{0i0} & \cdots & \sigma_{0ip} \\ \vdots & \ddots & \vdots \\ \sigma_{pi0} & \cdots & \sigma_{pip} \end{bmatrix};$$

$A_i$ and $B_i$ are the projections of $A(\theta)$ and $B(\theta)$ onto the i$th$ basis function $\phi_i$; $\sigma_{lmn} = \langle \phi_l, \phi_m, \phi_n \rangle / \langle \phi_l^2 \rangle$; and $e_a = [1, 0, \ldots, 0]^\top \in \mathbb{R}^a$.

The orthogonality property of the multivariate polynomial basis functions can now be used to efficiently compute the moments of the conditional probability distribution $\mathsf{P}[\hat{x}_{t+k|t}|w]$ in terms of the coefficients $\tilde{\mathbf{x}}_{t+k|t}$. The conditional mean and variance of the i$th$ predicted state are defined by

$$\mathbb{E}[\hat{x}_{i,t+k|t}|w] \approx \tilde{x}_{i_0,t+k|t}(w) \tag{20a}$$

$$\mathbb{E}\left[\hat{x}_{i,t+k|t}^2|w\right] \approx \sum_{j=0}^{p} \tilde{x}_{i_j,t+k|t}^2(w)\langle \phi_j^2 \rangle. \tag{20b}$$

Similarly, the state feedback control law (16) is projected as

$$\tilde{\mathbf{u}}_{t+k|t} = \mathbf{L}_{t+k|t}\tilde{\mathbf{x}}_{t+k|t} + \mathbf{m}_{t+k|t}, \tag{21}$$

where $\mathbf{L}_{t+k|t} = L_{t+k|t} \otimes I_{p+1}$ and $\mathbf{m}_{t+k|t} = m_{t+k|t} \otimes e_{p+1}$. When $w$ is a zero-mean Gaussian white noise with covariance $\Sigma_w$, $\tilde{\mathbf{x}}_{t+k|t}$ will be a Gaussian process with mean $\bar{\mathbf{x}}_{t+k|t}$ and covariance $\Gamma_{t+k|t}$ as defined by

$$\bar{\mathbf{x}}_{t+k+1|t} = (\mathbf{A} + \mathbf{B}\mathbf{L}_{t+k|t})\bar{\mathbf{x}}_{t+k|t} + \mathbf{B}\mathbf{m}_{t+k|t} \tag{22a}$$

$$\mathbf{\Sigma}_{t+k+1|t} = (\mathbf{A} + \mathbf{B}\mathbf{L}_{t+k|t})\mathbf{\Sigma}_{t+k|t}(\mathbf{A} + \mathbf{B}\mathbf{L}_{t+k|t})^\top + \mathbf{D}\mathbf{\Sigma}_w\mathbf{D}^\top. \tag{22b}$$

Note that $(\bar{\mathbf{x}}_{t|t}, \mathbf{\Sigma}_{t|t})$ is initialized based on the knowledge of state $x_t$. Using (20)-(22) and the law of iterated expectation, tractable expressions are derived for describing the mean and variance of each (polynomial chaos-approximated) state $\hat{x}_{i,t+k|t}$

$$\begin{aligned}\mathbb{E}[\hat{x}_{i,t+k|t}] &= \mathbb{E}\big[\mathbb{E}[\hat{x}_{i,t+k|t}|w]\big] \\ &\approx \mathbb{E}[\tilde{x}_{i_0,t+k|t}(w)] = \bar{x}_{i_0,t+k|t}\end{aligned} \tag{23}$$

and

$$\begin{aligned}&\mathbb{E}[\hat{x}^2_{i,t+k|t}] = \mathbb{E}\big[\mathbb{E}[\hat{x}^2_{i,t+k|t}|w]\big] \\ &\approx \mathbb{E}\big[\sum_{j=0}^{p} \tilde{x}^2_{i_j,t+k|t}(w)\langle\phi_j^2\rangle\big] = \sum_{j=0}^{p} \mathbb{E}[\tilde{x}^2_{i_j,t+k|t}]\langle\phi_j^2\rangle \\ &= \sum_{j=0}^{p} \big[\bar{x}_{i_j,t+k|t}^{\,2} + \mathbf{\Sigma}_{i_j i_j,t+k|t}\big]\langle\phi_j^2\rangle,\end{aligned} \tag{24}$$

respectively. It is important to note that the moments (23)-(24) can be expressed as explicit functions of the control inputs, i.e., the decision variables $\mathbf{L}$ and $\mathbf{m}$ in (22). The sample-free, closed-form expressions (23)-(24) for the moments of the predicted states are highly advantageous for gradient-based optimization methods since they avoid possible convergence problems associated with sampling.

### *C. Moment-based surrogate for joint chance constraint*

We look to replace the joint chance constraint (14) with a deterministic surrogate defined in terms of the mean and covariance of the predicted state $x_{t+k|t}$. Consider $x \sim \mathcal{N}(\bar{x}, \Sigma)$ as a $n_x$-dimensional multivariate Gaussian random vector and let $\mathscr{X} := \{\zeta : C^\top \zeta \leq d\}$. This allows for rewriting the joint chance constraint (14) as

$$\mathbb{P}(x \in \mathscr{X}) = \frac{1}{\sqrt{(2\pi)^{n_x}\det(\Sigma)}} \int_{\mathscr{X}} e^{-\frac{1}{2}(\zeta-\bar{x})^\top \Sigma^{-1}(\zeta-\bar{x})} d\zeta \;\; \geq \;\; 1-\delta. \tag{25}$$

To obtain a relaxation for (25), define the ellipsoid $\mathscr{E}_r := \{\zeta : \zeta^\top \Sigma^{-1} \zeta \leq r^2\}$ with radius $r$. Expression (25) is guaranteed to hold when

$$\bar{x} \oplus \mathscr{E}_r \subset \mathscr{X} \quad \implies \quad \mathbb{P}(x \in \mathscr{X}) > \mathbb{P}(x \in \bar{x} \oplus \mathscr{E}_r) = 1-\delta,$$

which indicates that the smallest radius of ellipsoid $\mathscr{E}_r$ must be chosen such that $\mathbb{P}(x \in \bar{x} \oplus \mathscr{E}_r) = 1-\delta$ [46]. Equivalently, $\mathbb{P}(x \in \bar{x} \oplus \mathscr{E}_r) = 1-\delta$ can be represented in terms of a chi-squared cumulative distribution function $F_{\chi^2_n}$ with $n$ degrees of freedom

$$\mathbb{P}(x \in \bar{x} \oplus \mathscr{E}_r) = \mathbb{P}\big((x-\bar{x})^\top \Sigma^{-1}(x-\bar{x}) \leq r^2\big) = F_{\chi^2_n}(r^2) = \frac{\gamma\left(\frac{n}{2}, \frac{r^2}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}, \tag{26}$$

where $\gamma$ is the lower incomplete Gamma function and $\Gamma$ is the complete Gamma function. This implies that the radius $r$ can be selected such that $F_{\chi^2_n}(r^2) = 1-\delta$ in order to guarantee that $\bar{x} \oplus \mathscr{E}_r \subset \mathscr{X}$ is satisfied. For the expression (26) to hold, the ellipsoid $\mathscr{E}_r$ must lie in the intersection of half-spaces $\mathscr{H}_j := \{\zeta : c_j^\top \zeta \leq d_j\}$, where $c_j \in \mathbb{R}^{n_x}$ is the jth column of $C$ and $d_j \in \mathbb{R}$ is the jth element of $d$. We use the result of the following lemma to derive an expression for guaranteeing the inclusion of $\mathscr{E}_r$ in the half-spaces $\mathscr{H}_j$, which relies on the Mahalanobis distance $d_M(x) = \sqrt{(x-\bar{x})^\top \Sigma^{-1}(x-\bar{x})}$ [30].

*Lemma 1:* The Mahalanobis distance to the hyperplane $h^\top x = g$ is given by $d_M(x^\star) = \frac{(g-h^\top \bar{x})}{\sqrt{h^\top \Sigma h}}$, where $x^\star = \bar{x} + \frac{(g-h^\top \bar{x})}{\sqrt{h^\top \Sigma h}} \delta x$ and $\delta x = \frac{\Sigma h}{\sqrt{h^\top \Sigma h}}$.

Lemma 1 indicates that $x^\star$ is the "worst-case" vector at which the ellipsoid $\mathscr{E}_r$ with radius $d_M(x^\star)$ intersects the hyperplane, while $\delta x$ is the direction along which $x^\star$ lies. Lemma 1 leads to the assertion that $\bar{x} \oplus \mathscr{E}_r \subset \mathscr{X}$ is equivalent to

$$\frac{(d_j - c_j^\top \bar{x})}{\sqrt{c_j^\top \Sigma c_j}} \geq r, \qquad j = 1, \dots, n_c. \tag{27}$$

Expression (27) results in an exact moment-based surrogate for the joint chance constraint (14)

$$c_j^\top \bar{x}_{t+k|t} + r\sqrt{c_j^\top \Sigma_{t+k|t} c_j} \leq d_j, \qquad j = 1, \dots, n_c, \tag{28}$$

where $\bar{x}_{t+k|t}$ and $\Sigma_{t+k|t}$ are, respectively, the mean and covariance of the predicted state $x_{t+k|t}$ in (13); and $r$ must satisfy $F_{\chi^2_{n_c}}(r^2) = 1-\delta$. The mean and covariance of the predicted state can be approximated in terms of the gPC-based moment expressions (23)-(24).

### *D. Sample-free, moment-based SMPC formulation*

We now present a sample-free formulation for SMPC of system (13). Using the gPC-based prediction model (19) and the input parameterization (21), the control cost function (15) can be (approximately) rewritten as

$$J_N(x_t, \mathbf{L}^\mathbf{N}, \mathbf{m}^\mathbf{N}) = \mathbb{E}_{x_t}\Big[\sum_{k=0}^{N-1} \|\tilde{\mathbf{x}}_{t+k|t}\|^2_{\mathbf{Q}} + \|\mathbf{L}_{t+k|t}\tilde{\mathbf{x}}_{t+k|t} + \mathbf{m}_{t+k|t}\|^2_{\mathbf{R}}\Big],$$

where $\mathbf{Q} = Q \otimes W$; $\mathbf{R} = R \otimes W$; $W = \mathrm{diag}(\langle \phi_0^2 \rangle, \langle \phi_1^2 \rangle, \dots, \langle \phi_p^2 \rangle)$; and $\mathbf{L}^\mathbf{N}$ and $\mathbf{m}^\mathbf{N}$ are the vectors of decision variables over the prediction horizon $N$. The sample-free SMPC algorithm involves solving the following

OCP

$$
\begin{aligned}
&\min_{\mathbf{L}^{\mathbf{N}},\mathbf{m}^{\mathbf{N}}} \quad J_N(x_t,\mathbf{L}^{\mathbf{N}},\mathbf{m}^{\mathbf{N}}) \\
\text{s.t.:}\ & \bar{\mathbf{x}}_{t+k+1|t} = (\mathbf{A}+\mathbf{B}\mathbf{L}_{t+k|t})\bar{\mathbf{x}}_{t+k|t} + \mathbf{B}\mathbf{m}_{t+k|t}, && k=0,\dots,N-1, \\
& \boldsymbol{\Sigma}_{t+k+1|t} = (\mathbf{A}+\mathbf{B}\mathbf{L}_{t+k|t})\boldsymbol{\Sigma}_{t+k|t}(\mathbf{A}+\mathbf{B}\mathbf{L}_{t+k|t})^\top + \mathbf{D}\boldsymbol{\Sigma}_w\mathbf{D}^\top, && k=0,\dots,N-1, \\
& c_j^\top \mathbb{E}[\hat{x}_{t+k|t}] + r\sqrt{c_j^\top\left(\mathbb{E}[\hat{x}_{t+k|t}\hat{x}_{t+k|t}^\top] - \mathbb{E}[\hat{x}_{t+k|t}]\mathbb{E}[\hat{x}_{t+k|t}]^\top\right)c_j} \le d_j, && \forall j,\ k=1,\dots,N, \\
& \mathbf{L}_{t+k|t}\tilde{\mathbf{x}}_{t+k|t} + \mathbf{m}_{t+k|t} \in \mathbb{U}, && k=0,\dots,N-1.
\end{aligned}
$$

The prediction model in the above OCP describes the evolution of the mean $\bar{\mathbf{x}}_{t+k|t}$ and covariance $\boldsymbol{\Sigma}_{t+k|t}$ of the coefficients of the PCEs of the states, $\tilde{\mathbf{x}}_{t+k|t}$, over the prediction horizon. The prediction model is initialized using the knowledge of the true state $x_t$ at each sampling time $t$. The surrogate for the joint chance constraint is defined in terms of the mean and covariance of the polynomial chaos-approximated states $\hat{x}_{t+k|t}$, which are computed in terms of the mean $\bar{\mathbf{x}}_{t+k|t}$ and covariance $\boldsymbol{\Sigma}_{t+k|t}$ using the expressions (23)-(24).

### *E. Extensions*

A limitation of gPC is the ability to handle correlated random variables. When the random variables $\xi$ are correlated, the PCE (17) is not guaranteed to converge. An alternative to gPC, termed arbitrary polynomial chaos (aPC), that can address this shortcoming has recently been presented [37]. APC allows for constructing a set of orthogonal polynomial basis in terms of the raw moments of the uncertainties using a multivariate generalization of the Gram-Schmidt process [35]. APC holds promise for devising efficient, sample-free algorithms for stochastic optimization and SMPC of systems with correlated probability uncertainty.

## V. Stochastic tube MPC

Tube MPC for control of deterministic systems under uncertainty has been developed in [32], [31], [40], [41], [42]. Stochastic tube MPC approaches have been proposed in [12], [14], [28], [13] and described in the book [26]. The approach in [28] exploits linear models affected by stochastic disturbances without assuming that disturbance values are normally distributed. This approach is further discussed in this Section and illustrated with an example.

### A. System model, disturbance model and constraints

The control design exploits a linear prediction model of the form,

$$x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t} + Dw_{t+k|t}, \tag{30}$$

where $x_{t+k|t} \in \mathbb{R}^{n_x}$ and $u_{t+k|t} \in \mathbb{R}^{n_u}$ are the predicted state and control sequences $k$ steps ahead starting from the current time $t$, and the elements of the disturbance sequence, $w_{t+k|t} \in \mathbb{R}^{n_w}$, are assumed to be zero mean, identically distributed, and independent for different $k$. Furthermore, these disturbance values are compactly-supported,

$$w_{t+k|t} \in \Pi = \{w: \ |w_i| \leq \alpha_i, \ i = 1, \cdots, n_w\}. \tag{31}$$

Probabilistic chance constraints are imposed as

$$\mathsf{P}[y_{t+k|t} = Cx_{t+k|t} \leq y_{max}] \geq 1 - \varepsilon, \tag{32}$$

where, to simplify the exposition, the case of the scalar output, $y \in \mathbb{R}^1$, is considered.

### B. Tube MPC design

The tube MPC controller is formed as a combination of the nominal state feedback and manipulatable input adjusted by the MPC controller,

$$u_{t+k|t} = Kx_{t+k|t} + g_{t+k|t}, \tag{33}$$

where the matrix $\Phi = (A + BK)$ is Schur and the sequence $g_{t+k|t}$, $k = 0, \cdots, N-1$, is optimized, with $g_{k|t} = 0$ for $k \geq N$. The receding horizon implementation involves using the first element of the optimized sequence, $g^*_{t|t}$, leading to a feedback law,

$$u_t = u_{MPC}(x_t) = Kx_t + g^*_{t|t}, \tag{34}$$

where $x_t$ is the current state at the time instance $t$ and $u_t$ is the control input at $t$.

The linearity of the prediction model permits to decompose the predicted state based on the superposition principle as

$$x_{t+k|t} = z_{t+k|t} + e_{t+k|t}, \tag{35}$$

where $z_{t+k|t}$ is the state prediction based on the nominal system,

$$z_{t+k+1|t} = \Phi z_{t+k|t} + Bg_{t+k|t}, \tag{36}$$

and $e_{t+k|t}$ is the error induced by the disturbance, given by

$$e_{t+k+1|t} = \Phi e_{t+k|t} + Dw_{t+k|t}. \tag{37}$$

Tube MPC approaches generally proceed by steering the state of the nominal system (36) with tightened constraints to account for the contributions of the error system (37).

The constraint (32) imposed over the prediction horizon can now be re-stated as

$$C[\Phi^{k-1}B \;\; \Phi^{k-2}B \;\cdots\; B \;\; 0\cdots 0]\mathbb{G}_t + C\Phi^k z_t \leq y_{max} - \gamma_k,\; k = 1,2,\cdots \tag{38}$$

where

$$Prob\{C[\Phi^{k-1}D \;\; \Phi^{k-2}D \;\cdots\; D \;\; 0\cdots 0]\mathbb{W}_t \leq \gamma_k\} = 1-\varepsilon, \tag{39}$$

and

$$\mathbb{G}_t = \begin{bmatrix} g_{t|t} \\ g_{t+1|t} \\ \vdots \\ g_{t+N-1|t} \end{bmatrix}, \quad \mathbb{W}_t = \begin{bmatrix} w_{t|t} \\ w_{t+1|t} \\ \vdots \\ w_{t+N-1|t} \end{bmatrix}.$$

The computation of $\gamma_k$ in (39) requires constructing the distribution of

$$C(\Phi^{k-1}Dw_{t|t} + \cdots + Dw_{t+k-1|t}),$$

for $t = 0$ (since problem characteristics are time-invariant). This can be performed offline by numerical approximation of the convolution integrals or by random sampling methods. In the random sampling approach, $N_w$ disturbance sequence scenarios are generated and the smallest number $\tilde{\gamma}_k$ is found such that $N^*/N_w \geq 1-\varepsilon$, where $N^*$ is the number of sequences for which $C(\Phi^{k-1}Dw_{t|t} + \cdots + Dw_{t+k-1|t}) \leq \tilde{\gamma}_k$. As $N_w \to \infty$, we expect that $\tilde{\gamma}_k \to \gamma_k$.

In the case of $w_{k|t}$ being independent and identically distributed, the Chebyshev inequality can be used to replace $\gamma_k$ with bounds based on

$$\gamma_k \leq \kappa\sqrt{CP_kC^{\mathrm{T}}},\; \kappa^2 = \frac{1-\varepsilon}{\varepsilon},\; P_{k+1} = \Phi P_k \Phi^{\mathrm{T}} + D\mathbb{E}[ww^{\mathrm{T}}]D^T,\; P_0 = 0,\; k = 0,1,2,\cdots. \tag{40}$$

To guarantee recursive feasibility, the constraint (38) is tightened to ensure that a feasible extension of $\mathbb{G}_t$ of the type “shift by 1 and pad by 0” exists at time $t+1$. This can be assured, as the disturbance takes values in a compact set, by the following “worst-case” constraints:

$$C[\Phi^{k-1}B \;\; \Phi^{k-2}B \;\cdots\; B \;\; 0\cdots 0]\mathbb{G}_t + C\Phi^k z_t \leq y_{max} - \beta_k,\; k = 1,2,\cdots, \tag{41}$$

$$\beta_k = \max\{\gamma_k, \gamma_{k-1} + a_{k-1}, \gamma_{k-2} + a_{k-2} + a_{k-1}, \cdots, \gamma_1 + a_1 + \cdots, a_{k-1}, 0\}, \tag{42}$$

$$a_k = \max_{w\in\Pi} C\Phi^k Dw.$$

The sequence $\{\beta_k\}$ is monotonically nondecreasing and upper bounded by a computable upper bound.

Minimizing the cost function defined by

$$J_N = \mathbb{E}\sum_{k=0}^{N-1}[x_{t+k|t}^\top Q x_{t+k|t} + u_{t+k|t}^\top R u_{t+k|t}] + \mathbb{E}[x_{t+N|t}^\top P x_{t+N|t}], \tag{43}$$

is replaced equivalently by minimizing

$$\tilde{J}_N = \sum_{k=0}^{N-1}[z_{t+k|t}^\top Q z_{t+k|t} + u_{t+k|t}^\top R u_{t+k|t}] + z_{t+N|t}^\top P z_{t+N|t},$$

which is a quadratic function of $\mathbb{G}_t$. Hence the following optimization problem is solved

$$\mathbb{G}_t^* = \arg\min_{\mathbb{G}_t} \tilde{J},$$

subject to

$$C[\Phi^{k-1}B \;\; \Phi^{k-2}B \;\cdots\; B \;\; 0\cdots 0]\mathbb{G}_t + C\Phi^k z_t \le y_{max} - \beta_k,\; k = 1,2,\cdots,N,$$

$$z_{N|t} \in S,$$

where $S$ is a finitely determined inner approximation of the maximum output admissible set defined by constraints,

$$\{z_N :\; C\Phi^j z_N \le y_{max} - \beta_{N+j}, j = 0,1,\cdots\}.$$

The terminal constraint set can be constructed as

$$\begin{aligned} S = \{z_N :\; & C\Phi^j z_N \le y_{max} - \beta_{N+j}, j = 1,\cdots,\hat{N}, \\ & C\Phi^l z_N \le y_{max} - \bar{\beta},\; l = \hat{N}+1,\cdots,\hat{N}+n^*\}, \end{aligned}$$

where $n^*$ must be sufficiently large, and

$$\bar{\beta} = \gamma_1 + \sum_{j=1}^{\bar{N}} a_j$$

for $\bar{N}$ sufficiently large.

### *C. Theoretical guarantees*

Theoretical guarantees for the closed-loop behavior under the tube SMPC law are available for the case when $\tilde{J}_N$ in (43) is modified to an infinite prediction horizon cost (the control horizon is still $N$),

$$\tilde{J}_N = \mathbb{E}\sum_{k=0}^{\infty}[x_{t+k|t}^\top Q x_{t+k|t} + u_{t+k|t}^\top R u_{t+k|t} - l_{ss}], \tag{44}$$

where

$$l_{ss} = \lim_{k\to\infty} \mathbb{E}(x_{t+k|t}^\top Q x_{t+k|t} + u_{t+k|t}^\top R u_{t+k|t}),$$

is the steady-state value of the stage cost under the control $u_{t+k|t} = Kx_{t+k|t}$. This value can be computed as

$$l_{ss} = trace(\Theta(Q + K^\top RK)), \quad \Theta - \Phi\Theta\Phi^\top = D\mathbb{E}[ww^\top]D^\top.$$

Note that (44) is a quadratic function of $\mathbb{G}_t$ (the details of the computations of this function are given in Chapter 6 of [26]). Then, under suitable assumptions, given feasibility at the time instant $t = 0$, the problem remains feasible at all future time instants, and causes the closed loop system to satisfy the probabilistic constraint (39) and the quadratic stability condition,

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=0}^{t} \mathbb{E}\left[x_k^\top Qx_k + u_k^\top Ru_k\right] \leq l_{ss}.$$

### *D. Mass-spring-damper example*

We consider a mass-spring-damper example with the model and constraint given by

$$m\ddot{x}_1 + c\dot{x}_1 + kx_1 = w - u, \tag{45}$$

$$y = x_1 \leq y_{max}, \tag{46}$$

with $m = 1$, $c = 0.1$, $k = 7$, and $y_{max} = 1$. The model is converted to the form (30) by defining $x = [x_1,\ x_2]^\mathrm{T}$ and using the sampling period, $\Delta T = 0.1$ sec. The disturbance force samples, $w_t$, are assumed to be distributed according to the truncated Gaussian distribution, with zero mean, standard deviation of $\frac{1}{12}$ and truncation interval $[-0.2, 0.2]$.

Two methods to compute $\gamma_k$ were considered, one based on random sampling and the other one based on Chebyshev's inequality. Over a 1000 simulated trajectories, a constraint violation rate metric was defined, as the maximum over $t$ of the fraction of trajectories violating the constraints at the instant $t$. See Figure 2. The trajectories for $\varepsilon = 0.2$, corresponding to 80% confidence of constraint satisfaction, are shown in Figure 3.

### *E. Extensions*

The above approach utilizes the so-called polytopic stochastic tubes. Stochastic tubes with ellipsoidal cross-section to bound $e_{k|t}$ with probability at least $1-\varepsilon$ can also be used. Polytopic tubes can be extended to handle both additive and multiplicative uncertainties. Typical assumptions involve

$$(A(q), B(q), w(q)) = (A^{(0)}, B^{(0)}, 0) + \sum_{j=1}^{m} (A^{(j)}, B^{(j)}, w^j)q^{(j)}$$

where $q^{(j)}$ are scalar random variables and $q_t = [q_t^{(1)}, q_t^{(2)}, \cdots, q_t^{(m)}]^\top$ are independent for different time instants, identically distributed and have known probability distribution.

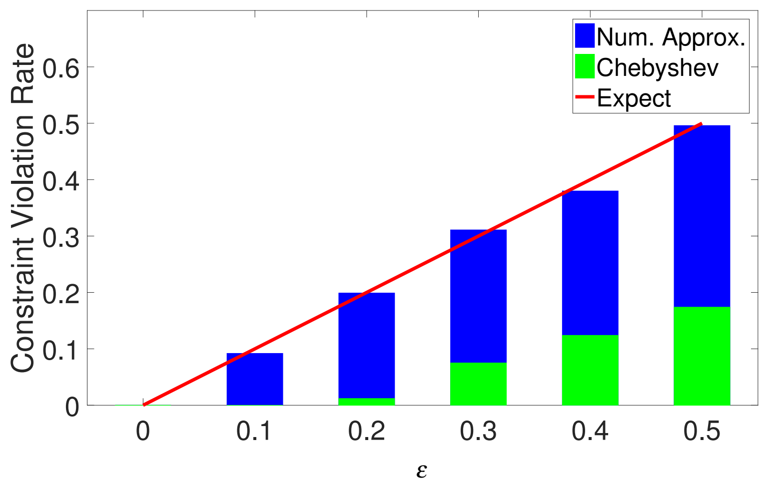


Fig. 2. Expected and estimated rate of constraint violations.

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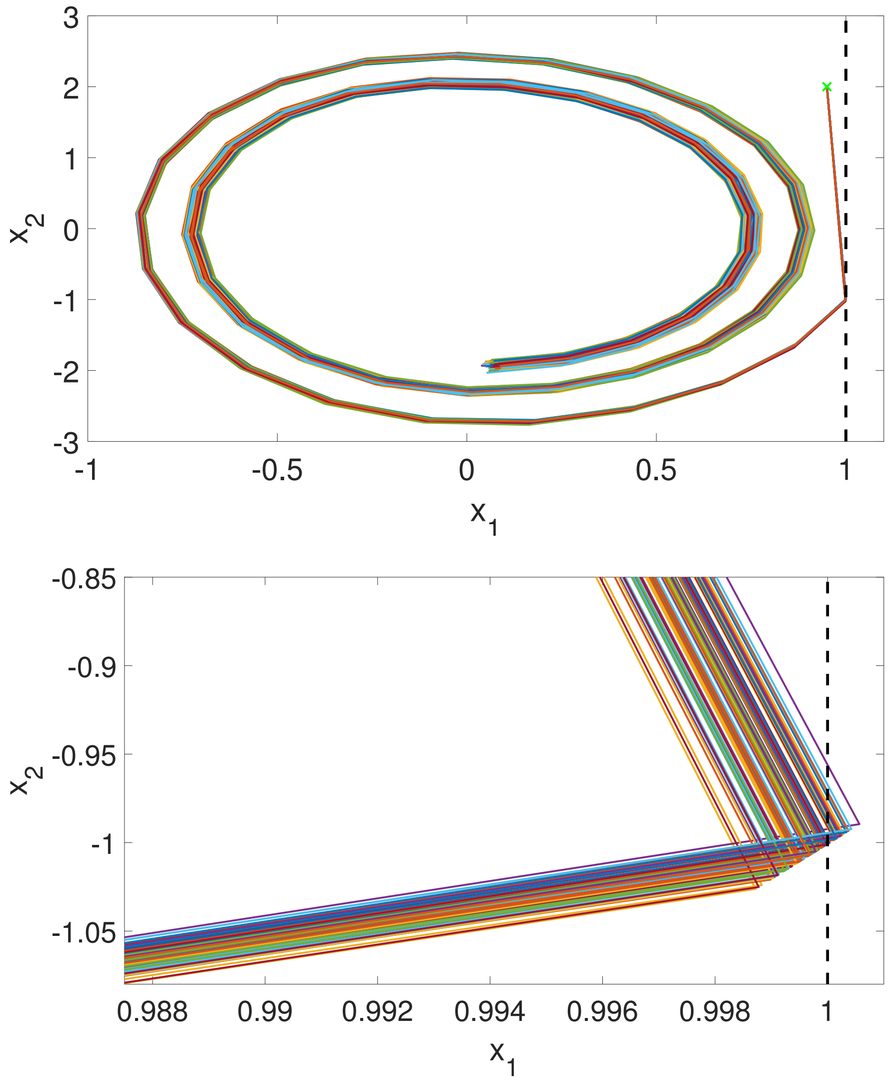


Fig. 3. Trajectories on $x_1$-$x_2$ plane for $\varepsilon = 0.2$, bottom plot: zoomed-in. Constraint is shown by the dashed vertical line.


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